Numerical Integration of Partial Differential Equations

Tutorial on **FIDISOL/CADSOL**

Jorge Delgado

22^{nd} and 24th of November 2022

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The solver can provide error estimates for the computed solution.

Newton-Raphson Method

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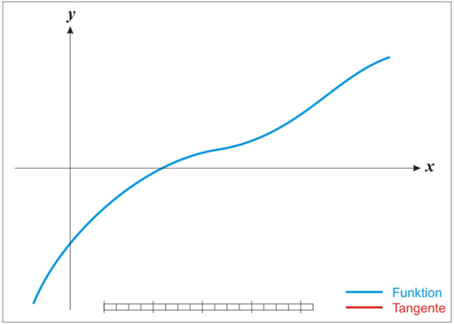
The Newton-Raphson method is a method to find roots of functions, by means of their derivatives.

By providing a good initial guess, it is possible to obtain the root iteratively through,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Wikipedia: Newton's method

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- A good initial guess for the solution must be given.
- The user has to provide boundary conditions, as well as a mesh for (x, y) with $N_x \times N_y$ points.

Numerical Approach

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2. Introduce an improve solution,

$$u^{(2)} = u^{(1)} + s\,\Delta u^{(1)} \tag{1}$$

   where s is a relaxation factor (usually equal to 1), and suppose that the new improve solution solves the system, *i.e.*, $P(u^{(2)}) = 0$.

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3. Expand $P(u^{(2)})$ in the small parameter $\Delta u^{(1)}$ to first order,

$$0 = P(u^{(2)}) = P(u^{(1)} + s\ \Delta u^{(1)}) \approx P(u^{(1)}) + \frac{\partial P}{\partial u}(u^{(1)})\Delta u^{(1)} + \dots$$

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4. Compute $\Delta u^{(1)}$ through the above equation and obtain the new improve solution $u^{(2)}$ using Eq. 1.
5. Repeat iteratively the last 3 steps until the Newton residual $P(u^{(N)})$ is lower than a prescribed tolerance.

Error Estimation

FIDISOL/CADSOL can provide error estimates for the solution.

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  The discretization is done by using the backward finite difference method.

Other errors estimates can be computed using well-know black hole physics results, such as, the Smarr Law or the 1$^{\text{st}}$ Law of Thermodynamics.

Kerr Solution

The Kerr solution is one of the most relevant solution in General Relativity. It can be write, in Boyer-Lindquist coordinate, as,

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\varphi\right)^2 + \Sigma\left(\frac{dr^2}{\Sigma} + d\theta^2\right) + \frac{\sin^2\theta}{\Sigma}\left[adt - \left(\Sigma + a^2\cos^2\theta\right)\right]^2$$

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What if we did not had an analytical close form of the Kerr solution?

How would we obtain the Kerr metric?

Ansatz Metric

The *ansatz* metric which is suitable for our problem is,

$$ds^2 = -e^{2F_0} N dt^2 + e^{2F_1}\left(\frac{dr^2}{N} + r^2 d\theta^2\right) + e^{2F_2} r^2 \sin^2\theta \left(d\varphi - W dt\right)^2 \ , \quad N = 1 - \frac{r_H}{r}$$

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→ F_1, F_2, F_0 and W are *ansatz* function that depend on $\{r, \theta\}$.

→ r_H is the radial coordinate of the event horizon and is an input parameter of the problem.

Connection between Metrics

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This non-standard coordinate system relates to the Boyer-Lindquist coordinates by a radial transformation,

$$r = R - \frac{a^2}{R_H}$$

→ R and R_H is the radial coordinate and radius of the event horizon in Boyer-Lindquist coordinates.

→ $a = J/M$ is the reduced angular momentum.

Connection between Metrics

The corresponding expressions for the metric functions read,

$$e^{2F_1} = \left(1 - \frac{c_t}{r}\right)^2 + c_t(c_t - r_H)\frac{\cos^2\theta}{r^2}$$

$$e^{2F_2} = e^{-2F_1}\left\{\left[\left(1 - \frac{c_t}{r}\right)^2 + \frac{c_t(c_t - r_H)}{r^2}\right]^2 + c_t(r_H - c_t)\left(1 - \frac{r_H}{r}\right)\frac{\sin^2\theta}{r^2}\right\}$$

$$F_0 = -F_2\,, \quad W = e^{-2(F_1+F_2)}\sqrt{c_t(c_t - r_H)}\frac{r_H - 2c_t}{r^3}\left(1 - \frac{c_t}{r}\right)$$

→ $c_t < 0$ is a constant that does not have a transparent meaning, but is related to the non-stationary of the solution.

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Several quantities of interest can be computed,

$$M = \frac{1}{2}(r_H - 2c_t)\,, \quad J = \frac{1}{2}\sqrt{c_t(c_t - r_H)}(r_H - 2c_t)\,, \quad A_H = 4\pi(r_H - c_t)(r_H - 2c_t)$$

$$T_H = \frac{r_H}{4\pi(r_H - c_t)(r_H - 2c_t)}\,, \quad \Omega_H = \frac{\sqrt{c_t(c_t - r_H)}}{(r_H - c_t)(r_H - 2c_t)}$$

Boundary Conditions

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  The absence of conical singularities impose that, on the poles, $F_1 = F_2$.
  Furthermore, due to the symmetry on the equatorial plane, we only compute everything on the region $\theta \in \left[0, \frac{\pi}{2}\right]$. Therefore, we also impose the following boundary conditions at the equatorial plane,

$$\partial_\theta F_i = \partial_\theta W = 0\,, \ \text{at} \ \theta = \frac{\pi}{2}$$

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$$F_i = {F_i}^{(0)} + x^2 {F_i}^{(2)} + \mathcal{O}(x^4)$$
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  It is natural now to impose the following boundary conditions at the horizon,

$$\partial_x F_i = 0\,, \quad W = \Omega_H\,, \quad \text{at} \quad r = r_H$$

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Vacuum Einstein equations,

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To solve these equations, we will use everything we talked so far, including the radial coordinate transformation $x = \sqrt{r^2 - r_H^2}$. Thus, we have to use a new version of the *ansatz* metric,

$$ds^2 = -e^{2F_0} x^2 \frac{H(x)}{S(x)} + e^{2F_1} \left(\frac{dx^2}{H(x)} + S(x) d\theta^2 \right) + e^{2F_2} S(x) \sin^2 (d\varphi - W dt)^2$$

$$S(x) = x^2 + {r_H}^2 \text{ and } H(x) = \frac{\sqrt{S(x)}}{r_H + \sqrt{S(x)}}$$

Let's go to Mathematica!

Combination of Einstein Equations

FIDISOL/CADSOL likes to solve system of PDEs where each one has only second derivatives of a single functions. Thus, we have to combine the Einstein equations in the following way,

$$
\begin{aligned}
&-G_t^t + G_r^r + G_\theta^\theta - G_\varphi^\varphi = 0 \\
&G_t^t + G_r^r + G_\theta^\theta - G_\varphi^\varphi + 2WG_\varphi^t = 0 \\
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\end{aligned}
$$

We can define two other equations that will serve as constrains and will be used to test the numerical accuracy of the solutions,

$$
\begin{aligned}
&G_r^r - G_\theta^\theta = 0 \\
&G_r^\theta = 0
\end{aligned}
$$

Boundary Conditions Revisited

- Asymptotic boundary conditions: $\lim_{r\to\infty} F_i = \lim_{r\to\infty} W = 0$
- Axis boundary conditions: $\partial_\theta F_i = \partial_\theta W = 0\,,\ \text{ at }\ \theta = \{0, \pi\}$
- Equatorial boundary conditions: $\partial_\theta F_i = \partial_\theta W = 0\,,\ \text{ at }\ \theta = \frac{\pi}{2}$
- Event Horizon boundary conditions: $\partial_x F_i = 0\,,\ \ W = \Omega_H\,,\ \text{ at }\ r = r_H$

Radial Coordinate Transformation

Unfortunately, computers do not like infinite, and our radial coordinate x ranges from 0 to ∞. Thus, a compactification is required. Such is done by,

$$\bar{x} = \frac{x}{1 + x}$$

The transformation implies a substitution of the derivatives of the *ansatz* functions,

$$\partial_x \mathcal{F} \longrightarrow (1 - \bar{x})^2 \, \partial_{\bar{x}} \mathcal{F} \, , \quad \partial_{xx} \mathcal{F} \longrightarrow (1 - \bar{x})^4 \, \partial_{\bar{x}\bar{x}} \mathcal{F} - 2 \, (1 - \bar{x})^3 \, \partial_{\bar{x}} \mathcal{F}$$

Newton Residual for the Boundary Conditions

Take as an example the boundary conditions at the horizon,

$$\partial_x F_i = 0\,, \quad W = \Omega_H\,, \quad \text{at} \quad r = r_H$$

Those boundaries can be rewritten as,

$$\partial_x F_i = 0\,, \quad W - \Omega_H = 0\,, \quad \text{at} \quad r = r_H$$

This way, the solver just has to find the roots of those boundaries, which is exactly what **FIDISOL/CADSOL** is good at!

Run FIDISOL/CADSOL

For Linux/macOS,

1. Compile the file "2-CAD.f" with the flags "-c" and "-O2",
   - ifort -c -O2 2-CAD.f
2. Compile the file "v1.for" with the same flags,
   - ifort -c -O2 v1.for
3. Compile together both outputs of the above codes without flags,
   - ifort 2-CAD.o v1.o -o runKerr
4. Run the executable "runKerr".

For Windows,

1. Compile the file "2-CAD.f" with the flags "/nolink" and "/O2",
   - ifort /nolink /O2 2-CAD.f
2. Compile the file "v1.for" with the same flags,
   - ifort /nolink /O2 v1.for
3. Compile together both outputs of the above codes without flags,
   - ifort 2-CAD.obj v1.obj /exe runKerr
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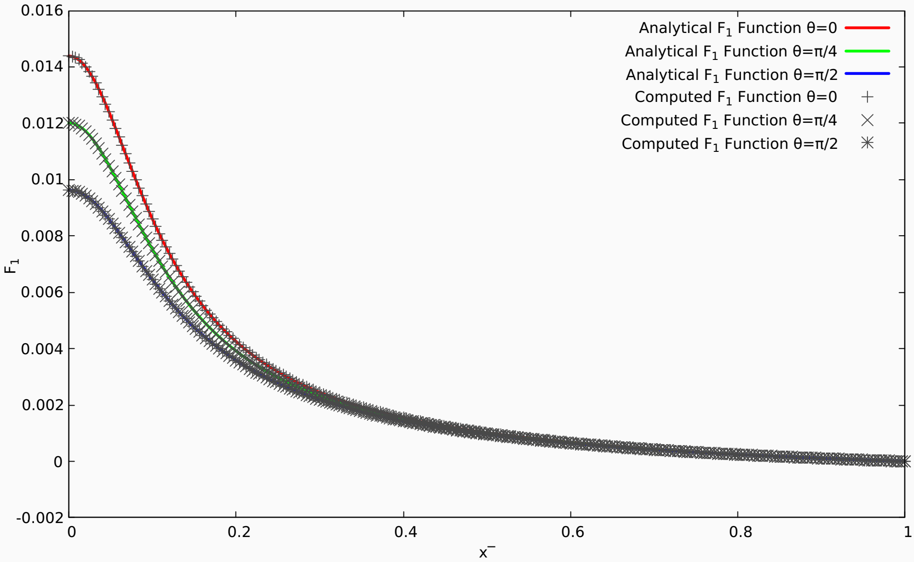
Comparison Exact Kerr with Computed Kerr



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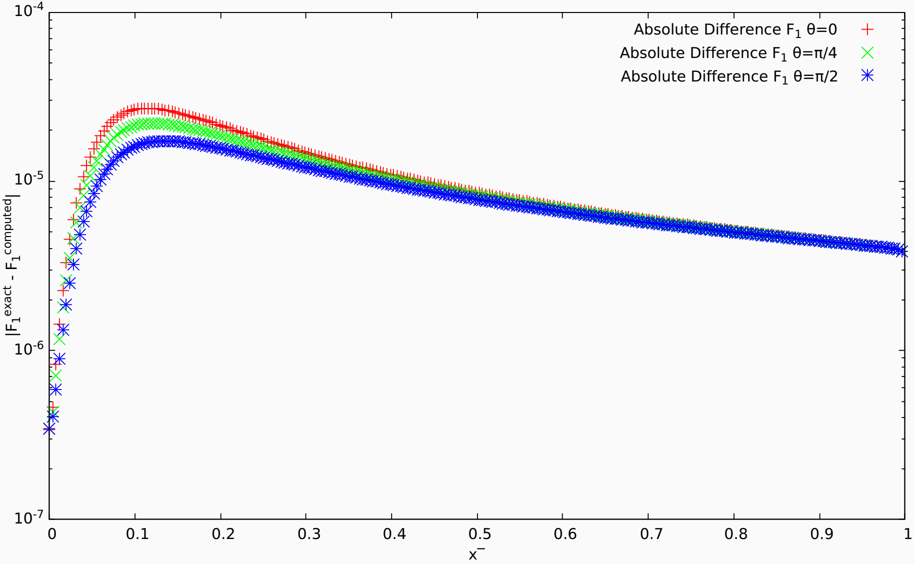

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Jorge Delgado

22nd and 24th of November 2022

Boson Stars

Boson Stars

Boson stars are horizonless, self-gravitating, solitonic-like, scalar field solutions of the Einstein-Klein-Gordon action,

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi} - g^{\mu\nu}\partial_\mu\Psi^*\partial_\nu\Psi - V(|\Psi|^2)\right]$$

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$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - g^{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi - V(|\Psi|^2) \right]$$

The potential has the form,

$$V(|\Psi|^2) = \mu^2 |\Psi|^2 + \dots$$

→ μ is the mass of the field.

→ "…"correspond to higher orders terms of the scalar field, *e.g.* self-interactions.

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These changes, introduces a new *ansatz* metric,

$$ds^2 = -e^{2F_0} dt^2 + e^{2F_1} \left(dr^2 + r^2 d\theta^2\right) + e^{2F_2} r^2 \sin^2\theta \left(d\varphi - \frac{W}{r} dt\right)^2$$

$$\Psi = \phi \, e^{-i(\omega t - m\varphi)}$$

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$$\Psi = \phi\; e^{-i(\omega t - m\varphi)}$$

→ ϕ is a new *ansatz* function that depends on $\{r, \theta\}$

→ ω is the angular frequency of the scalar field.

→ $m = \pm\{1, 2, \ldots\}$ is the azimuthal harmonic index.

Boundary Conditions for ϕ

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- Asymptotic boundary condition: Asymptotically flatness implies that,

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The equations of motion are now composed by the Einstein equations and the Klein-Gordon equation,

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To perform the numerical integration, it is useful and convenient to use dimensionless variables. such as,

$$r \to \mu r \,, \quad \omega \to \omega/\mu \,, \quad \phi \to \phi/\sqrt{4\pi}$$

Let's go our dear friend Mathematica!

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Mass and Angular Momentum from Asymptotic Behaviour

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The mass and angular momentum of the boson star can be obtained through the asymptotic behaviour of the metric functions,

$$g_{tt} = -e^{2F_0} + e^{2F_2} W^2 \sin^2\theta = -1 + \frac{2M}{r} + \dots\,, \quad g_{t\varphi} = -e^{2F_2} W r \sin^2\theta = -\frac{2J}{r}\sin^2\theta + \dots$$

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Asymptotically, we have,

$$F_0 = -\frac{M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \quad W = \frac{2J}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

With a simple linearization in $1/r$ and $1/r^2$, we obtain the mass and angular momentum, respectively.

Mass and Angular Momentum from the Komar Integrals

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Komar integrals are defined as,

$$M = -2 \int_{\Sigma} dS_{\mu} \left(T^{\mu}_{\nu} t^{\nu} - \frac{1}{2} T t^{\mu} \right) \,, \quad J = \int_{\Sigma} dS_{\mu} \left(T^{\mu}_{\nu} \varphi^{\nu} - \frac{1}{2} T \varphi^{\mu} \right)$$

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Expanding the integrals,

$$M = -\int_0^{\infty} dr \int_0^{\pi} d\theta \ r^2 \sin\theta \ e^{F_0 + 2F_1 + F_2} \left(T^t_t - \frac{1}{2} T \right)$$

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   Perimeteral radius is a radial coordinate R, such that a circumference along the equatorial plane has perimeter $2\pi R$, and relates to r by $R = e^{F_2} r$.

2. Define the inverse compactness as the ratio between R_{99} and the Schwarzschild radius associated with a 99% of the boson star mass, $R_{Schw} = 2M_{99}$

$$\text{Compactness}^{-1} = \frac{R_{99}}{2M_{99}}$$

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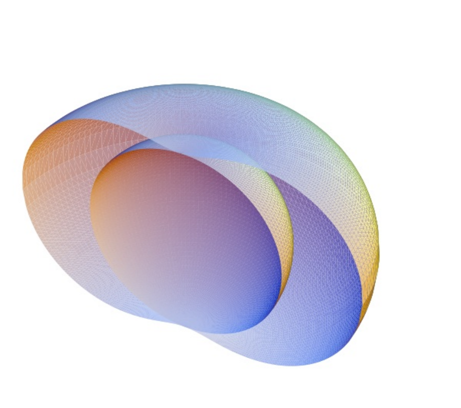

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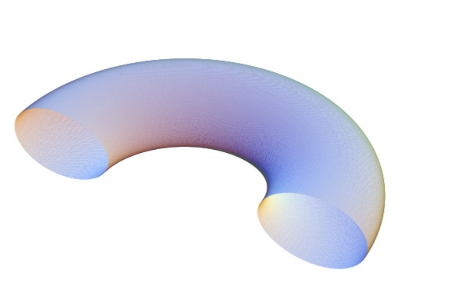

Light rings

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1. Start with the effective Lagrangian for geodesic motion of a massless test particle, on the equatorial plane,
$$\mathcal{L} = e^{2F_1}\dot{r}^2 + e^{2F_2}r^2\left(\dot{\varphi} - \frac{W}{r}\dot{t}\right)^2 - e^{2F_0}\dot{t}^2 = 0$$

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3. Rewrite the Lagrangian and obtain an equation for $\dot{r}^2$,
$$\dot{r}^2 = V(r) = e^{2F_1}\left[e^{-2F_0}\left(E - L\frac{W}{r}\right)^2 - e^{-2F_2}\frac{L^2}{r^2}\right]$$

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   In the end, we just have to find the roots of the following equations,

$$e^{F_0}\left[1 - r\left(F_0{}' - F_2{}'\right)\right] \pm e^{F_2}\left(W - rW'\right) = 0$$

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