# Series Expansions of the Equations of Motion at the Boundaries - Kerr

## Expansion around the horizon $x = 0$

### Clear Variables

### Input Equations of Motion

### Computation

In[ ]:=
```
H[x_] := Sqrt[x^2 + rH^2] / (rH + Sqrt[x^2 + rH^2]);
S[x_] := rH^2 + x^2;
```

$$H[x\_] := \frac{\sqrt{x^2 + rH^2}}{rH + \sqrt{x^2 + rH^2}};$$

$$S[x\_] := rH^2 + x^2;$$

In[ ]:=
```
F0[x_, θ_] := f0[θ] + f0x[θ] x + f0xx[θ] x^2;
F1[x_, θ_] := f1[θ] + f1x[θ] x + f1xx[θ] x^2;
F2[x_, θ_] := f2[θ] + f2x[θ] x + f2xx[θ] x^2;
W[x_, θ_] := w[θ] + wx[θ] x + wxx[θ] x^2;
```

#### $0^{th}$ Order Approximation

In[ ]:=
```
Simplify[Series[x Eq1, {x, 0, 0}], rH > 0]
```

Out[ ]=

$$-\frac{1}{2} e^{-2\,f0[\theta]+2\,f2[\theta]}\,rH^2\,Sin[\theta]^4 \left(rH^2\,wx[\theta]^2 + 2\,w'[\theta]^2\right) + O[x]^1$$

In[ ]:=
```
Simplify[Series[x^2 Eq2, {x, 0, 0}], rH > 0]
```

Out[ ]=

$$e^{-2\,f0[\theta]+2\,f2[\theta]}\,rH^2\,Sin[\theta]^4 \left(rH^2\,wx[\theta]^2 + 2\,w'[\theta]^2\right) + O[x]^1$$

In[ ]:=
```
Simplify[Series[x Eq3, {x, 0, 0}], rH > 0]
```

Out[ ]=

$$-2 \left(e^{-2\,f0[\theta]+2\,f2[\theta]}\,rH^2\,Sin[\theta]^3 \left(rH^2\,wx[\theta]^2 + 2\,w'[\theta]^2\right)\right) + O[x]^1$$

In[∘]:= `Simplify[Series[Eq4, {x, 0, 0}], rH > 0]`

Out[∘]=

$O[x]^1$

In[∘]:= `Simplify[Series[x^2 ConstrainEq1, {x, 0, 0}], rH > 0]`

Out[∘]=

$$\frac{1}{2} e^{-2 (f0[\theta]+f1[\theta]-f2[\theta])} rH\, Sin[\theta]^2 \left(rH^3\, wx[\theta]^2 - 2\, rH\, w'[\theta]^2\right) + O[x]^1$$

In[∘]:= `Simplify[Series[x^2 ConstrainEq2, {x, 0, 0}], rH > 0]`

Out[∘]=

$$e^{-2 (f0[\theta]+f1[\theta]-f2[\theta])} rH^2\, Sin[\theta]^2\, wx[\theta]\, w'[\theta] + O[x]^1$$

Conclusion:

In[∘]:=
```
w[θ_] := ΩH;
wx[θ_] := 0;
```

## $1^{st}$ Order Approximation

In[∘]:= `Simplify[Series[x Eq1, {x, 0, 1}], rH > 0]`

Out[∘]=

$$-f2x[\theta]\, Sin[\theta]^2\, x + O[x]^2$$

In[∘]:= `Simplify[Series[x^2 Eq2, {x, 0, 1}], rH > 0]`

Out[∘]=

$$f2x[\theta]\, Sin[\theta]^2\, x + O[x]^2$$

In[∘]:= `Simplify[Series[x Eq3, {x, 0, 1}], rH > 0]`

Out[∘]=

$$2 \left(2\, f0x[\theta] + f2x[\theta]\right) Sin[\theta]\, x + O[x]^2$$

In[∘]:= `Simplify[Series[Eq4, {x, 0, 1}], rH > 0]`

Out[∘]=

$O[x]^3$

In[∘]:= `Simplify[Series[x^2 ConstrainEq1, {x, 0, 1}], rH > 0]`

Out[∘]=

$$e^{-2\, f1[\theta]} (-f0x[\theta] + f1x[\theta])\, x + O[x]^2$$

In[∘]:= `Simplify[Series[x^2 ConstrainEq2, {x, 0, 1}], rH > 0]`

Out[∘]=

$$\frac{e^{-2\, f1[\theta]} (-f0'[\theta] + f1'[\theta])\, x}{rH^2} + O[x]^2$$

Conclusion:

In[∘]:=
```
f0[θ_] := f1[θ] + c;
f2x[θ_] := 0;
f0x[θ_] := 0;
f1x[θ_] := 0;
```

### In the end

In[∘]:=
```
F0[r, θ]
```
Out[∘]=

$c + r^2\, f0xx[\theta] + f1[\theta]$

In[∘]:=
```
F1[r, θ]
```
Out[∘]=

$f1[\theta] + r^2\, f1xx[\theta]$

In[∘]:=
```
F2[r, θ]
```
Out[∘]=

$f2[\theta] + r^2\, f2xx[\theta]$

In[∘]:=
```
W[r, θ]
```
Out[∘]=

$\Omega H + r^2\, wxx[\theta]$

---

# Expansion on the poles $\theta = 0$

## Clear Variables

## Input Equations of Motion

## Computation

In[∘]:=
```
F0[x_, θ_] := f0[x] + f0θ[x] θ + f0θθ[x] θ^2;
F1[x_, θ_] := f1[x] + f1θ[x] θ + f1θθ[x] θ^2;
F2[x_, θ_] := f2[x] + f2θ[x] θ + f2θθ[x] θ^2;
W[x_, θ_] := w[x] + wθ[x] θ + wθθ[x] θ^2;
```

### $0^{th}$ Order Approximation

In[∘]:=
```
Simplify[Series[Eq1, {θ, 0, 0}]]
```
Out[∘]=

$O[\theta]^1$

In[ ]:=
```
Simplify[Series[Eq2, {θ, 0, 0}], rH > 0]
```
Out[ ]=

$$O[\theta]^1$$

In[ ]:=
```
Simplify[Series[Eq3, {θ, 0, 0}], rH > 0]
```
Out[ ]=

$$\frac{2\,x\,\mathrm{f0}\theta[x]}{H[x]\times S[x]} + O[\theta]^1$$

In[ ]:=
```
Simplify[Series[Eq4, {θ, 0, 0}], rH > 0]
```
Out[ ]=

$$O[\theta]^1$$

In[ ]:=
```
Simplify[Series[θ ConstrainEq1, {θ, 0, 0}], rH > 0]
```
Out[ ]=

$$-\frac{2\left(e^{-2\,\mathrm{f1}[x]}\left(\mathrm{f1}\theta[x]-\mathrm{f2}\theta[x]\right)\right)}{S[x]} + O[\theta]^1$$

In[ ]:=
```
Simplify[Series[θ ConstrainEq2, {θ, 0, 0}], rH > 0]
```
Out[ ]=

$$\frac{e^{-2\,\mathrm{f1}[x]}\left(\mathrm{f1}'[x]-\mathrm{f2}'[x]\right)}{S[x]} + O[\theta]^1$$

Conclusion:

In[ ]:=
```
f0θ[x_] := 0
f2θ[x_] := f1θ[x]
f1[x_] := f2[x] + c
```

## 1st Order Approximation

In[ ]:=
```
Simplify[Series[Eq1, {θ, 0, 1}]]
```
Out[ ]=

$$O[\theta]^2$$

In[ ]:=
```
Simplify[Series[Eq2, {θ, 0, 1}], rH > 0]
```
Out[ ]=

$$\frac{2\,\mathrm{f1}\theta[x]\,\theta}{H[x]\times S[x]} + O[\theta]^2$$

In[∘]:=
```
Simplify[Series[Eq3, {θ, 0, 1}], rH > 0]
```

Out[∘]=

$$\frac{1}{2\,H[x]\,S[x]^2}\Big(16\,x\,f0\theta\theta[x]\times S[x]+S[x]\Big(-x\,H'[x]\,S'[x]+2\,S[x]\Big(\big(3+3\,x\,f0'[x]+x\,f2'[x]\big)\,H'[x]+x\,H''[x]\Big)\Big)+H[x]\Big(x\,S'[x]^2+4\,S[x]^2\,\big(x\,f0'[x]^2+f2'[x]+f0'[x]\,\big(2+x\,f2'[x]\big)+x\,f0''[x]\big)-2\,x\,S[x]\,\big(f2'[x]\,S'[x]+S''[x]\big)\Big)\Big)\,\theta+O[\theta]^2$$

In[∘]:=
```
Simplify[Series[Eq4, {θ, 0, 1}], rH > 0]
```

Out[∘]=

$$\frac{3\,x^2\,w\theta[x]\,\theta}{H[x]\times S[x]}+O[\theta]^2$$

In[∘]:=
```
Simplify[Series[θ ConstrainEq1, {θ, 0, 1}], rH > 0]
```

Out[∘]=

$$\frac{1}{4\,x\,S[x]^2}\,e^{-2\,(c+f2[x])}\Big(-2\,x\,H[x]\,S'[x]^2+S[x]\,\big(-4\,x+8\,x\,f0\theta\theta[x]-4\,x\,f1\theta[x]^2-16\,x\,f1\theta\theta[x]+24\,x\,f2\theta\theta[x]+6\,H[x]\,S'[x]+6\,x\,H[x]\,f0'[x]\,S'[x]-2\,x\,H[x]\,f2'[x]\,S'[x]+3\,x\,H'[x]\,S'[x]\big)-2\,S[x]^2\,\Big(\big(3+3\,x\,f0'[x]-x\,f2'[x]\big)\,H'[x]+2\,H[x]\,\big(x\,f0'[x]^2-2\,f2'[x]-x\,f2'[x]^2+f0'[x]\,\big(2-2\,x\,f2'[x]\big)+x\,(f0''[x]+f2''[x])\big)+x\,H''[x]\Big)\Big)\,\theta+O[\theta]^2$$

In[∘]:=
```
Simplify[Series[θ ConstrainEq2, {θ, 0, 1}], rH > 0]
```

Out[∘]=

$$\frac{e^{-2\,(c+f2[x])}\,\Big(-2\,x\,H[x]\,f1\theta'[x]+f1\theta[x]\,\Big(2\,H[x]\,\big(1+x\,f0'[x]+x\,f2'[x]\big)+x\,H'[x]\Big)\Big)\,\theta}{2\,x\,H[x]\times S[x]}+O[\theta]^2$$

Conclusion:

In[∘]:=
```
f1θ[x_] := 0
wθ[x_] := 0
```

## In the end

In[∘]:=
```
F0[r, θ]
```

Out[∘]=

$$f0[r]+\theta^2\,f0\theta\theta[r]$$

In[∘]:=
```
F1[r, θ]
```

Out[∘]=

$$c+\theta^2\,f1\theta\theta[r]+f2[r]$$

In[∘]:=
```
F2[r, θ]
```

Out[∘]=

$$f2[r]+\theta^2\,f2\theta\theta[r]$$

In[ ]:= **W[r, θ]**

Out[ ]=

$w[r] + \theta^2\, w\theta\theta[r]$