A Simple two-dimensional model for the Hawking radiation

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Introduction

A black hole: the object formed, e.g., by the gravitational collapse of a large star.

In classical physics nothing comes out of a black hole. Even light cannot escape from it.

Hawking discovered more than 50 years ago that *in quantum field theory in curved spacetime* a black hole of mass M radiates like a black body of temperature, called the Hawking temperature T_{H},

$$\boxed{k_B T_{\mathsf{H}} = \frac{\hbar c^3}{8\pi G M}}$$

$k_B = 1.38 \times 10^{-23}$ J/K: Boltzmann constant
$\hbar = 1.054 \times 10^{-34}$ J·sec: Planck constant
$G = 6.67 \times 10^{-11}$m^3/kg·sec^{-2}: Newton constant
$c = 3.00 \times 10^8$m/sec: Light Speed constant
For a spherically symmetric black hole with $M = 30M_\odot$ ($M_\odot = 1.99 \times 10^{30}$kg): $T_{\mathsf{H}} = 6.17 \times 10^{-8}$K.

Hawking's original derivation is difficult

Hawking derived his radiation for a spherically symmetric black hole.

It involves sophisticated use of mathematics in quantum field theory in curved spacetime and unreasonable idealization.

The spacetime is time-dependent:

almost flat spacetime $\rightarrow$ black-hole spacetime

difficult to treat exactly

Question: Is there a simple model allowing exact calculations but still capturing the essence of the Hawking effect?

Perfectly reflecting collapsing spherical mirror

Quantum field: massless scalar (i.e., spin-0) field

- Perfectly reflecting spherical mirror of mass M at $r = r_0 > 2GM/c^2$. ($r = 2GM/c^2$ would be the black-hole horizon after the collapsing.)
- The scalar field in the vacuum state without any radiation coming in or going out (the Boulware vacuum)
- The mirror starts collapsing at $t = 0$, in a spherically symmetric way, and disappears below $r = 2GM/c^2$.

Perfectly reflecting collapsing spherical mirror

- The spacetime outside the mirror is a static spherically symmetric spacetime with black hole of mass M (Schwarzschild spacetime) throughout the mirror-collapsing process because of Birkoff's Theorem.
- The problem reduces to finding out the evolution of the state with a reflecting boundary at $r = R(t)$ (radius of the mirror) with the initial condition at $t = 0$ being the Boulware vacuum, which is the no-particle state.

Still complicated mainly due to scattering by the spacetime curvature.

In two dimensions, there is no scattering for massless scalar particles in any spacetime.

We study a two-dimensional analogue of QFT of massless scalar field bounded by the perfectly reflecting collapsing mirror, which is a point, with the state of the scalar field which being the Boulware vacuum state at $t = 0$.

The plan

(1) Quick review of (2D) Schwarzschild spacetime in Kruskal-Szekeres coordinates

(2) Massless scalar field in two dimensions: ingoing and outgoing fields

(3) Renormalized stress-energy tensor

(4) Renormalized stress-energy tensor in the Boulware and Hartle-Hawking vacua

(5) Renormalized stress-energy tensor in the perfectly reflecting collapsing mirror: energy flux consistent with a thermal bath of temperature $\kappa/2\pi$, $\kappa = 1/4M$ $(G = c = \hbar = T_B = 1)$, is seen to go out to infinity.

Apology: It will NOT be shown that the energy distribution is thermal. It will be shown only that the total flux equals that of a thermal bath of temperature $\kappa/2\pi$.

(2D) Schwarzschild spacetime

U: const. on the out-going light ray: describes the out-going field
V: const. on the in-going light ray: describes the in-going field.

$$d\tau^2 = \frac{32M^3}{r} e^{-r/2M} dU dV\,, \quad UV = -\frac{r-2M}{2M} e^{r/2M}\,,\; -V/U = e^{t/2M}\,.$$

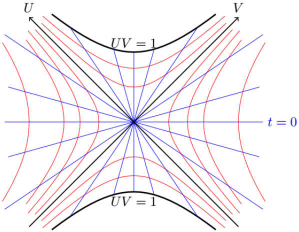
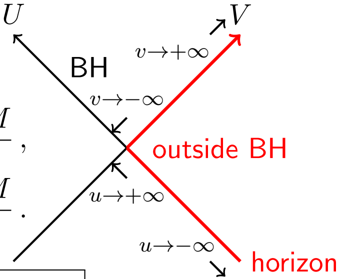


Figure: The Schwarzschild black hole in Kruskal-Szekeres coordinates U and V. the red curves: constant-r hypersurfaces; the blue lines: constant-t hypersurfaces. The singularities: $UV = 1$ ($r = 0$). Horizons ($r = 2M$) are at $U = 0$ and $V = 0$.

(2D) Schwarzschild spacetime

black-hole region: $U, V > 0$.
Outside region: $U < 0$,
$V > 0$. Let $\boxed{U = -e^{-\kappa u}}$,
$\boxed{V = e^{\kappa v}}$ $(\kappa = 1/4M)$.



$$u = t - r_* = t - r - 2M \log \frac{r - 2M}{2M},$$

$$v = t + r_* = t + r + 2M \log \frac{r - 2M}{2M}.$$

$\boxed{d\tau^2 = C(u, v) du dv, \ C(u, v) = 1 - 2M/r}$.
Eddington-Finkelstein coordinates.
The variable u plays the role of time for the out-going field.
As $u \to \infty$ (infinite future), $U \to 0$ (future horizon).

Massless scalar field

$d\tau^2 = G(u, v) du dv$

The field equation for the massless scalar field $\phi(u, v)$:

$$\nabla_\mu \nabla^\mu \phi = 0 \Rightarrow \partial_u \partial_v \phi = 0 \,, \textit{ for any } G(u, v).$$

General solution: $\phi(u, v) = \phi^{(o)}(u) + \phi^{(i)}(v)$.
$\phi^{(o)}(u)$: the outgoing field $(u = t - r_*)$
$\phi^{(i)}(v)$: the ingoing field $(v = t + r_*)$.

We mainly discuss only the outgoing field: $f(u)$ is an outgoing wave solution for any f.

Massless scalar field in Schwarzschild spacetime

The quantized outgoing field

$$\widehat{\phi^{(o)}}(u) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[\hat{a}(\omega)e^{-i\omega u} + \hat{a}^\dagger(\omega)e^{i\omega u}\right] ,$$
$$[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega') .$$

$e^{-i\omega u} = e^{-i\omega t}e^{i\omega r_*} = e^{-i\omega t}[(r-2M)/2M]^{i\omega/2M}e^{i\omega r}$

The Boulware vacuum $|0_{\rm B}\rangle$: $\hat{a}(\omega)|0_{\rm B}\rangle = 0$ for all ω.

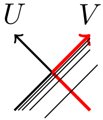
No-particle state with respect to the usual energy

$U = -e^{-\kappa u}$, $\kappa = 1/4M$ (surface gravity).

$U \to 0^-$ as $u \to \infty$.

Near $U = 0$,

$e^{-i\omega u}$ oscillates infinitely many times.



Massless scalar field in Schwarzschild spacetime

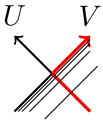
Global Schwarzschild spacetime: $d\tau^2 = K(U,V)dUdV$.
$(U = -e^{-\kappa u},\ V = e^{\kappa v})$

$$\partial_U \partial_V \phi = 0 \Rightarrow \phi(U,V) = \phi^{(O)}(U) + \phi^{(I)}(V)\,.$$

The quantized outgoing field

$$\widehat{\phi^{(O)}}(U) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[\hat{A}(\omega)e^{-i\omega U} + \hat{A}^\dagger(\omega)e^{i\omega U}\right]$$

$$[\hat{A}(\omega), \hat{A}^\dagger(\omega')] = \delta(\omega - \omega')\,.$$



The Hartle-Hawking vacuum $|0_{\mathrm{HH}}\rangle$:
$\hat{A}(\omega)|0_{\mathrm{HH}}\rangle = 0$ for all ω.

We'll find that $|0_{\mathrm{HH}}\rangle$ contains energy fluxes consistent with a thermal bath of temperature $\kappa/2\pi$ relative to $|0_{\mathrm{B}}\rangle$ by studying the (renormalized) stress-energy tensor.

The stress-energy tensor

For the massless scalar field

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \phi)(\nabla^\alpha \phi) \, .$$

For the spacetime with $d\tau^2 = C(u,v) du dv$,

$$T_{uu} = (\partial_u \phi)^2 = (\partial_u \phi^{(o)}(u))^2 \, , \; T_{vv} = (\partial_v \phi)^2 = (\partial_v \phi^{(i)}(v))^2 \, , \; T_{uv} = 0 \, .$$

The trace: $g^{\mu\nu} T_{\mu\nu} = 2 g^{uv} T_{uv} = 0$. A consequence of the scale invariance

T_{uu}: the outgoing energy flux
T_{vv}: the ingoing energy flux

The stress-energy tensor

$$\widehat{\phi^{(o)}}(u) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}[\hat{a}(\omega)e^{-i\omega u} + \hat{a}^\dagger(\omega)e^{i\omega u}]\,,$$
$$[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega')\,.$$

Assume that the state $|\psi\rangle$ satisfies $\langle\psi|\hat{a}(\omega)\hat{a}(\omega')|\psi\rangle = 0$.

$$\langle\psi|T_{uu}|\psi\rangle = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}}\omega\omega'$$
$$\times\,[\langle\psi|\hat{a}^\dagger(\omega)\hat{a}(\omega')|\psi\rangle e^{i(\omega-\omega')u} + \langle\psi|\hat{a}(\omega)\hat{a}^\dagger(\omega')|\psi\rangle e^{-i(\omega-\omega')u}]$$
$$= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}} 2\omega\omega'\langle\psi|\hat{a}^\dagger(\omega)\hat{a}(\omega')|\psi\rangle e^{i(\omega-\omega')u}$$
$$+ \frac{1}{4\pi}\int_0^\infty \omega\, d\omega\,.$$

The renormalized stress-energy tensor

For (two-dimensional) Minkowski spacetime $d\tau^2 = dUdV$,
$U = T - X$, $V = T + X$.
We renormalized the stress-energy tensor by normal ordering
$\hat{A}(\omega)\hat{A}^\dagger(\omega) \to \hat{A}^\dagger(\omega)\hat{A}(\omega)$:

$$\langle\psi|T_{UU}^{\text{ren}}|\psi\rangle = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}} 2\omega\omega' \langle\psi|\hat{A}^\dagger(\omega)\hat{A}(\omega')|\psi\rangle e^{i(\omega-\omega')U} .$$

Minkowski vacuum $|0_\text{M}\rangle$: $\hat{A}(\omega)|0_\text{M}\rangle = 0 \Rightarrow \langle 0_\text{M}|T_{UU}^{\text{ren}}|0_\text{M}\rangle = 0.$

Thermal state $|\kappa\rangle$ with temperature $\kappa/2\pi$ $\left(e^{\hbar\omega/k_B(\kappa/2\pi)} \to e^{2\pi\omega/\kappa}\right)$:
$\langle\kappa|\hat{A}^\dagger(\omega)\hat{A}(\omega')|\kappa\rangle = (e^{2\pi\omega/\kappa} - 1)^{-1}\delta(\omega - \omega')$:

$$\boxed{\langle\kappa|T_{UU}^{\text{ren}}|\kappa\rangle = \int_0^\infty \frac{d\omega}{4\pi\omega} \frac{2\omega^2}{e^{2\pi\omega/\kappa} - 1} = \frac{\kappa^2}{48\pi}}$$

Renormalized stress-energy tensor

For general spacetime with $d\tau^2 = C(u,v)dudv$:

$$\widehat{\phi}(u) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}[\hat{a}(\omega)e^{-i\omega u} + \hat{a}^\dagger(\omega)e^{i\omega u}]\,.$$

The conformal vacuum $|0_C\rangle$ for conformal factor $C(u,v)$ is defined by $\hat{a}(\omega)|0_C\rangle = 0$ for all ω.

It turns out to be inconsistent to let $\langle 0_C|T^{\text{ren}}_{uu}|0_C\rangle = 0$ (by normal ordering). We use "point-splitting" to define $\langle 0_C|T^{\text{ren}}_{uu}|0_C\rangle$.

$$\langle 0_C|[\partial_u\widehat{\phi}(u_1)][\partial_u\widehat{\phi}(u_2)]|0_C\rangle = -\frac{1}{4\pi(u_1-u_2)^2}\,.$$

We subtract the flat-space equivalent and take the limit $u_1, u_2 \to u$ in order to define $\langle 0_C|T^{\text{ren}}_{uu}|0_C\rangle$.

Renormalized stress-energy tensor

$d\tau^2 = C(u,v)dudv$

Define on the line with v fixed,

$$s(u,v) = \int_{u_0}^{u} C(\tilde{u},v)d\tilde{u} \;\Rightarrow\; \frac{\partial s(u,v)}{\partial u} = C(u,v)\,.\text{(affine parameter)}$$

The flat-space equivalent on the v=constant line turns out to be

$$\langle 0|T_{s_1 s_2}|0\rangle = \langle 0|\partial_s\widehat{\phi}(u_1)\partial_s\widehat{\phi}(u_2)|0\rangle = -\frac{1}{4\pi[s(u_1,v)-s(u_2,v)]^2}\,.$$

$$\langle 0|[\partial_u\widehat{\phi}(u_1)][\partial_u\widehat{\phi}(u_2)]|0\rangle = -\frac{(\partial s/\partial u|_{u=u_1})(\partial s/\partial u|_{u=u_2})}{4\pi[s(u_1,v)-s(u_2,v)]^2}\,.$$

$$\langle 0_C|T^{\text{ren}}_{uu}|0_C\rangle \xleftarrow[u_1,u_2\to u]{} -\frac{1}{4\pi(u_1-u_2)^2} + \frac{(\partial s/\partial u|_{u=u_1})(\partial s/\partial u|_{u=u_2})}{4\pi[s(u_1,v)-s(u_2,v)]^2}\,.$$

Renormalized stress-energy tensor

Then we find

$$\langle 0_C | T^{\text{ren}}_{uu} | 0_C \rangle = -\frac{1}{12\pi} C^{1/2} \partial_u^2 C^{-1/2} \,,$$
$$\langle 0_C | T^{\text{ren}}_{vv} | 0_C \rangle = -\frac{1}{12\pi} C^{1/2} \partial_v^2 C^{-1/2} \,.$$

Then, it would be inconsistent with $\nabla^\mu T^{\text{ren}}_{\mu\nu} = 0$ if $\langle 0_C | T^{\text{ren}}_{uv} | 0_C \rangle = 0$.

The unique solution is

$$T^{\text{ren}}_{uv} = \frac{1}{12\pi} \left[C^{1/2} \partial_u \partial_v C^{-1/2} - C(\partial_u C^{-1/2})(\partial_v C^{-1/2}) \right] \,.$$

$$g^{\mu\nu} T^{\text{ren}}_{\mu\nu} = C^{-1} T_{uv} = \frac{1}{12\pi} R \qquad \text{(trace anomaly)}$$

The Boulware vacuum

$d\tau^2 = C(u,v)dudv$, $C = 1 - 2M/r$,
$(v-u)/2 = r + 2M\log[(r-2M)/2M]$.

$$\hat{\phi}(u) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}[\hat{a}(\omega)e^{-i\omega u} + \hat{a}^\dagger(\omega)e^{-i\omega u}]\,,$$
$$\hat{a}(\omega)|0_\mathsf{B}\rangle = 0\,.$$

$|0_\mathsf{B}\rangle$: Boulware vacuum.

$$\langle 0_\mathsf{B}|T_{uu}|0_\mathsf{B}\rangle = \langle 0_\mathsf{B}|T_{vv}|0_\mathsf{B}\rangle = \frac{1}{24\pi}\left(-\frac{M}{r^3} + \frac{3M^2}{2r^4}\right)$$
$$= \begin{cases} -\frac{\kappa^2}{48\pi} & \text{at } r = 2M\,, \\ 0 & \text{at } r = \infty. \end{cases}$$

General vacuum for the outgoing field

Let $\mathcal{U} = f(u)$ ($U = -e^{-\kappa u}$ is a special case) and

$$\widehat{\phi}(\mathcal{U}) = \int \frac{d\omega}{\sqrt{4\pi\omega}}[\hat{A}(\omega)e^{-i\omega\mathcal{U}} + \hat{A}^\dagger(\omega)e^{i\omega\mathcal{U}}],$$

and define $|0_f\rangle$ by $\hat{A}(\omega)|0_f\rangle = 0$ for all ω.

$d\tau^2 = C(u,v)dudv = C(u,v)[f'(u)]^{-1}d\mathcal{U}dv$: $|0_f\rangle$ is the conformal vacuum for conformal factor $C(u,v)[f'(u)]^{-1}$. We can find $\langle 0_f|T^{\text{ren}}_{\mathcal{U}\mathcal{U}}|0_f\rangle$ by using the general formula. By multiplying that by $(\partial\mathcal{U}/\partial u)^2$ we find

$$\langle 0_f|T^{\text{ren}}_{uu}|0_f\rangle = \langle T^{\text{ren}}_{uu}\rangle_{\text{B}} + \frac{1}{48\pi}\left\{3\frac{f'''(u)}{f'(u)} - 2\left[\frac{f''(u)}{f'(u)}\right]^2\right\}.$$

The Hartle-Hawking vacuum

$$\langle 0_f | T_{uu}^{\mathsf{ren}} | 0_f \rangle = \langle T_{uu}^{\mathsf{ren}} \rangle_{\mathsf{B}} + \frac{1}{48\pi} \left\{ 3 \frac{f'''(u)}{f'(u)} - 2 \left[\frac{f''(u)}{f'(u)} \right]^2 \right\} .$$

For the Hartle-Hawking vacuum $U = f(u) = -e^{-\kappa u}$.

$$\langle T_{uu}^{\mathsf{ren}} \rangle_{\mathsf{HH}} = \frac{1}{24\pi} \left(-\frac{M}{r^3} + \frac{3M^2}{2r^4} \right) + \frac{\kappa^2}{48\pi} , \quad \kappa = 1/4M .$$

Similarly,

$$\langle 0_{\mathsf{HH}} T_{vv}^{\mathsf{ren}} | 0_{\mathsf{HH}} \rangle = \frac{1}{24\pi} \left(-\frac{M}{r^3} + \frac{3M^2}{2r^4} \right) + \frac{\kappa^2}{48\pi} .$$

In the Hartle-Hawking vacuum, there are ingoing and outgoing energy fluxes consistent with the thermal fluxes with temperature $\kappa/2\pi$.

(2D) collapsing mirror

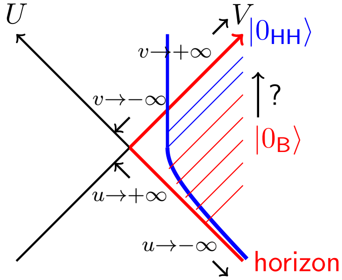
What happens if there is a static mirror reflecting the field at $r_0 > 2M$, the state being the Boulware vacuum, and if it starts collapsing at $t = 0$ and falls into the black hole?

2D collapsing mirror

The trajectory of the “mirror”: $r = r_0$ ($UV = -1$ or $V = -1/U$) before $(U, V) = (-1, 1)$ (or $(u, v) = (0, 0)$).

The Boulware vacuum is maintained by the static mirror.

The trajectory of the mirror is $V = F(U)$ with $F(0) = F'(0) > 0$ (guaranteeing that the mirror falls into the black hole) after $(U, V) = (-1, 1)$ (or $(u, v) = (0, 0)$.



2D collapsing mirror

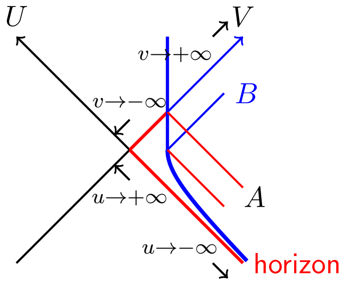
The in-going state $|0_{\mathrm{B}}\rangle$ at A:

$$\widehat{\phi}(v) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} [\hat{a}(\omega) e^{-i\omega v} + \hat{a}^\dagger(\omega) e^{i\omega v}] \, , \; \hat{a}(\omega)|0_{\mathrm{B}}\rangle = 0 \, .$$

The
variables u and v are mapped
to each other at the mirror as

$$V = F(U) \Rightarrow e^{\kappa v} = F(-e^{-\kappa u})$$

$$\Rightarrow v = f(u) = \frac{1}{\kappa} \log F(-e^{-\kappa u}) \, .$$



The outgoing
field at B is given by replacing
v by $f(u)$ and the state $|0_f\rangle$
is still defined by $\hat{a}(\omega)|0_f\rangle = 0$.

2D collapsing mirror

For $u > 0$, the state for the out-going field $|0_f\rangle$ satisfies $\hat{a}(\omega)|0_f\rangle = 0$, where

$$\widehat{\phi}(u) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}[e^{-i\omega\mathcal{U}}\hat{a}(\omega) + e^{-i\omega\mathcal{U}}\hat{a}^\dagger(\omega)e^{i\omega\mathcal{U}}]\,,$$

$$\mathcal{U} = f(u) = \frac{1}{\kappa}\log(F(U))\,, \quad U = -e^{-\kappa u}\,.$$

$$\langle 0_f|T^{\text{ren}}_{uu}|0_f\rangle = \langle 0_{\text{B}}|T^{\text{ren}}_{uu}|0_{\text{B}}\rangle + \frac{1}{48\pi}\left\{3\frac{f'''(u)}{f'(u)} - 2\left[\frac{f''(u)}{f'(u)}\right]^2\right\}\,.$$

$$\frac{dU}{du} = \kappa e^{-\kappa u} = -\kappa U\,.$$

$$f'(u) = -U\frac{F'(U)}{F(U)} \approx \frac{F'(0)}{F(0)}e^{-\kappa u} \quad \text{for large } u\,.$$

2D collapsing mirror

$$\langle 0_f | T^{\mathsf{ren}}_{uu} | 0_f \rangle = \langle 0_{\mathsf{B}} | T^{\mathsf{ren}}_{uu} | 0_{\mathsf{B}} \rangle + \frac{1}{48\pi} \left\{ 3 \frac{f'''(u)}{f'(u)} - 2 \left[\frac{f''(u)}{f'(u)} \right]^2 \right\} .$$

$$f'(u) = -U \frac{F'(U)}{F(U)} \approx \frac{F'(0)}{F(0)} e^{-\kappa u} \quad \text{for large } u \, .$$

$$f''(u) \approx -\kappa \frac{F'(0)}{F(0)} e^{-\kappa u} \, ,$$

$$f'''(u) \approx \kappa^2 \frac{F'(0)}{F(0)} e^{-\kappa u} \, .$$

Hence $\langle 0_f | T^{\mathsf{ren}}_{uu} | 0_f \rangle \approx \langle 0_{\mathsf{B}} | T^{\mathsf{ren}}_{uu} | 0_{\mathsf{B}} \rangle + \dfrac{\kappa^2}{48\pi}$ for large u.

Consistent with thermal flux of temperature $\kappa/2\pi$.

Summary

By considering a collapsing mirror in two-dimensional Schwarzschild spacetime, we have a simple model of the Hawking radiation with the out-going energy flux approaching to $\kappa^2/48\pi$ with $\kappa = 1/4M$, which is the energy flux for a thermal bath of temperature $\kappa/2\pi$. The transition time is $\sim 4M$ ($\approx 6 \times 10^{-4}$sec for $M = 30M_\odot$).

The energy flux going out to infinity is $\kappa^2/48\pi$ and that going into the horizon is $-\kappa^2/48\pi$ (same as in the Boulware vacuum). The collapsing mirror has little to do with the late-time Hawking radiation as far as the energy balance is concerned.

It will be interesting to examine the quantum state in more detail to find exactly what happens in the transition period.