An Analogue Model for a Particle in a Black Hole Background

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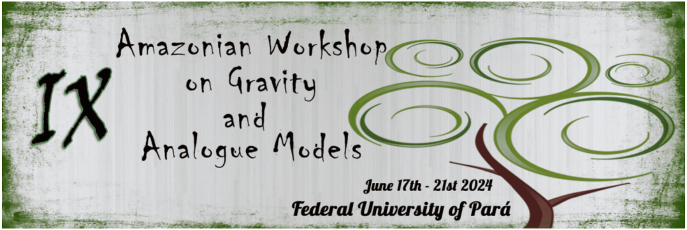

Motivation

- Some physical systems are naturally inaccessible to us;
- However, there are systems that can mimic the properties of those inaccessible systems — we call them *analogue models*;
- Examples: The magnetic monopole as target system in spin-ice systems, position-dependent mass quantum systems, Bose-Einstein condensates etc.

Objectives

- Solve the dynamics of a scalar particle in the background of a Kerr–de Sitter black hole;

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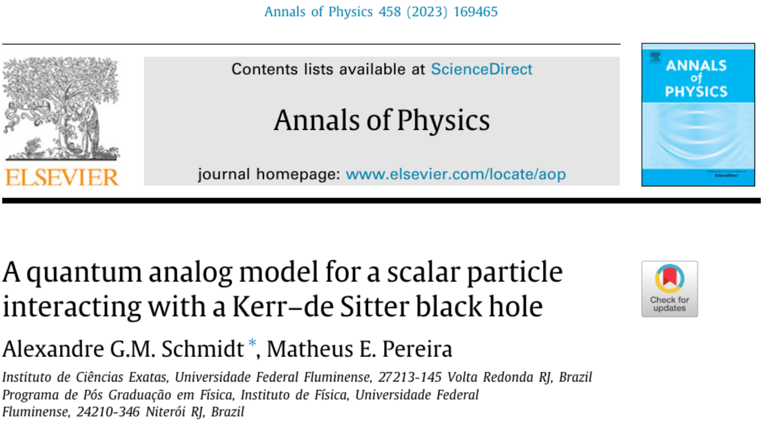
- Determine the angular and radial solutions of the problem;
- In parallel, we investigate the dynamics of a particle constrained to move on a spheroidal surface;
- We connect both problems using an external potential, thus creating an *analogue model*.



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A quantum analog model for a scalar particle interacting with a Kerr–de Sitter black hole

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Klein-Gordon Equation

The Kerr-de Sitter line element is χ, ξ, Ξ

$$ds^2 = -\frac{\Delta_\theta}{\Xi} D dt^2 + \frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2}\left[\frac{dr^2}{D} + \frac{r^2 + a^2}{\Delta_\theta} d\theta^2\right] + \frac{r^2 + a^2}{\Xi}\sin^2\theta d\phi^2$$

where $D = 1 - \frac{\Lambda r^2}{3}, \Delta_\theta = 1 + \frac{\Lambda a^2 \cos^2\theta}{3}, \Xi = 1 + \frac{\Lambda a^2}{3}$ and Λ is the cosmological constant.

Klein-Gordon Equation

The Klein-Gordon equation can be separated using $\psi(r,\theta,\phi,t) = F(r)G(\theta)\exp[i(m\phi+\omega t)]$. The radial part is

$$\frac{d}{dr}\left[(r^2+a^2)D\frac{dF}{dr}\right] + \left(\frac{3\Xi\omega^2}{\Lambda D} - \mu^2 r^2 + \frac{m^2 a^2 \Xi}{r^2+a^2} + \lambda\right) F = 0, \qquad (1)$$

and an angular equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\Delta_\theta\frac{dG}{d\theta}\right] - \left(\frac{3\Xi\omega^2}{\Lambda\Delta_\theta} + \mu^2 a^2\cos^2\theta + \frac{m^2\Xi}{\sin^2\theta} + \lambda\right) G = 0, \qquad (2)$$

Angular Klein-Gordon

We introduce $x = \cos^2\theta$ into the angular equation (2),

$$G'' + \left(\frac{1}{2x} + \frac{1}{x-1} + \frac{1}{x-a_H}\right)G' + \frac{1}{4x(x-1)\Delta_x}\left[\lambda - \frac{m^2\Xi}{x-1} + M^2a^2x - \frac{3\omega^2\Xi}{\Lambda\Delta_x}\right]G = 0, \quad (3)$$

with $a_H = -\frac{3}{a^2\Lambda}$ and $\Delta_x = 1 + \frac{\Lambda a^2 x}{3}$.

Angular Klein-Gordon

Applying a $F-$homotopic transformation

$$G(x) = x^{A}(x-1)^{B}(x-a_H)^{C}\, W(x), \tag{4}$$

with characteristic exponents equal to $A = \{0, \frac{1}{2}\}$, $B = \{\pm\frac{m}{2}\}$, and $C = \{\pm\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}\}$, we get a Heun differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a_H}\right)\frac{dy}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a_H)}y = 0, \tag{5}$$

Angular Klein-Gordon

$$G_{nm}^{(1)}(\theta) = (\cos^2\theta)^{1/2}(\cos^2\theta - 1)^{m/2}(\cos^2\theta - a_H)^C W_{nm}^{(1)}(\theta), \tag{6}$$

with

$$W_{nm}^{(1)}(\theta) = z^{1-\gamma}\left(\frac{z - a_H}{1 - a_H}\right)^{-\alpha-\beta+\gamma+\delta} H\ell\left(a_{new}, q_1; \alpha_1, \beta_1, \gamma_1, \delta_1; z\right),$$

and

$$G_{nm}^{(2)}(\theta) = (\cos^2\theta)^{1/2}(\cos^2\theta - 1)^{m/2}(\cos^2\theta - a_H)^C W_{nm}^{(2)}(\theta), \tag{7}$$

with

$$W_{nm}^{(2)}(\theta) = z^{1-\delta}\left(\frac{z - a_H}{1 - a_H}\right)^{-\alpha-\beta+\gamma+\delta} H\ell\left(a_{new}, q_2; \alpha_2, \beta_2, \gamma_2, \delta_2; z\right),$$

where $z = \sin^2\theta/(1 - a_H)$.

Radial Klein-Gordon

The radial eigenfunctions in the correct domain are

$$O_{nm}^{(1)}(r) = \left(\frac{u}{a_H}\right)^{1-\gamma_1} \left(\frac{u-1}{a_H-1}\right)^{-\beta_1+\gamma_1-1} H\ell\left(a_{new}, q_3, \alpha_3, \beta_3, \gamma_3, \delta_3; \frac{u-a_H}{a_H(u-1)}\right), \tag{8}$$

when the characteristic exponent is zero. For the other characteristic exponent $1-\varepsilon_1$ the eigenfunctions reads,

$$\begin{aligned} O_{nm}^{(2)}(r) &= \left(\frac{u}{a_H}\right)^{1-\gamma_1} \left(\frac{u-1}{a_H-1}\right)^{\alpha_1-\delta_1-1} (u-a_H)^{-\alpha_1-\beta_1+\gamma_1+\delta_1} \\ &\quad \times H\ell\left(a_{new}, q_4, \alpha_4, \beta_4, \gamma_4, \delta_4; \frac{u-a_H}{a_H(u-1)}\right), \end{aligned} \tag{9}$$

with $u = -r^2/a^2$.

Radial Klein-Gordon

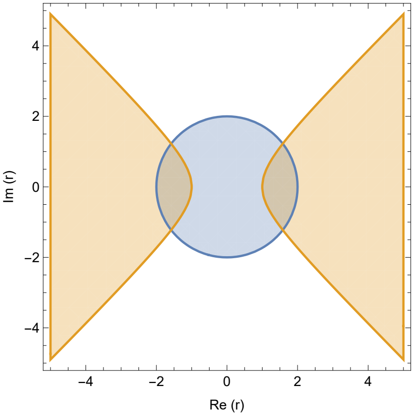

Figure: Complex $r-$plane. The eigenfunctions (8) and (9) converge whenever the variable r belongs to the yellow region defined by $\frac{u-a_H}{a_H(u-1)} < \min(1, |a_{new}|)$. Here $u = -r^2/a^2$, we choose $a = 2, \Lambda = 0.5$, so $a_H = -3/2$ and $a_{new} = -2/3$.

Thin–Layer

- The Thin–Layer Method allows us to constrain a particle on a curved surface;
- Such curved surfaces are feasible in a lab with current technology;
- The so called *geometric potential* binds the particle on the surface. It reads

$$V_g = \frac{-\hbar^2}{2m}(M^2 - K),$$

  where M and K are the mean and Gaussian curvatures;
- We can study dynamics of such particle and connect it with the dynamics of a particle in the black hole background.

Oblate Spheroid Coordinate System

The oblate spheroidal coordinates (η, θ, ϕ) are defined by,

$$\begin{cases} x' = a_0 \cosh\eta \sin\theta \cos\phi \\ y' = a_0 \cosh\eta \sin\theta \sin\phi \\ z' = a_0 \sinh\eta \cos\theta \end{cases}, \tag{10}$$

where $\eta > 0$ is a radial-like coordinate, $\pm a_0$ are the location of the foci, is $0 \leq \theta \leq \pi$ is the polar angle and $0 \leq \phi \leq 2\pi$ is the azimuthal angle.

Particle on an Oblate Spheroid

Following the thin-layer methodology [12, 13], Schrödinger equation is

$$
\begin{aligned}
-\frac{1}{2m^*}\Bigg[\frac{1}{a_0^2(s^2+\cos^2\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\
+\frac{1}{(s^2+1)\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Bigg]\psi + U\psi = E\psi,
\end{aligned}
\tag{11}
$$

The geometric potential reads,

$$
V_{geom}(\theta) = -\frac{s^2\sin^4\theta}{4a_0^2(1+s^2)(\cos^2\theta+s^2)^3}. \tag{12}
$$

Particle on an Oblate Spheroid

Upon substituting $\psi(\theta,\phi) = y(\theta)e^{im\phi}$, if we let $z = \cos^2\theta$ and $y(\sqrt{z}) = w(z)$, we can use

$$w(z) = z^A(z-1)^B\left(z+s^2\right)^C W(z). \tag{13}$$

The differential equation is transformed into,

$$w''(z) + \left(\frac{3}{2z} + \frac{1+m}{z-1} + \frac{2+\sqrt{5}}{2(z+s^2)}\right) w'(z) + \left(\frac{A_0}{z} + \frac{B_0}{z-1} + \frac{C_0}{z+s^2}\right) w(z) = 0, \tag{14}$$

Particle on an Oblate Spheroid

Comparing with equation (0.3) of Schäfke and Schmidt [29] — or with equation (120) of Kraniotis [30] — which is a generalized form of Heun's equation and reads,

$$y^{''}(z)+\left(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{z-1}+\frac{1-\mu_2}{z-a_H}+\alpha\right)y'(z)+\left[\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z(z-1)(z-a_H)}\right]y(z)=0, \tag{15}$$

Particle on an Oblate Spheroid

The first set of local solutions is given by,

$$y_{01}(z, \mu, \alpha, \lambda) = \eta(z, \mu_0, \mu_1, \mu_2, \alpha, \lambda; a_H), \tag{16}$$

and

$$y_{02}(z, \mu, \alpha, \lambda) = z^{\mu_0}\eta(z, -\mu_0, \mu_1, \mu_2, \alpha, \lambda; a_H), \tag{17}$$

where the Schäfke's $\eta-$function is defined by,

$$\eta(z, \mu, \alpha, \lambda; a_H) = \sum_{k=0}^{\infty} \frac{\tau_k(\mu, \alpha, \lambda; a_H) z^k}{\Gamma(1 + k - \mu_0)\Gamma(1 + k)}, \tag{18}$$

λ and μ stands for $(\lambda_0, \lambda_1, \lambda_2)$, and (μ_0, μ_1, μ_2), respectively.

Particle on an Oblate Spheroid

The first non-normalized solution reveals

$$\psi_{nm}^{(1)}(x) = (x^2)^{1/2}(x^2-1)^{m/2}\left(x^2+s^2\right)^{(2+\sqrt{5})/4}\eta\left(x^2,-\frac{1}{2},-m,-\frac{\sqrt{5}}{2},0,\lambda;-s^2\right), \tag{19}$$

and the second non-normalized solution is given by

$$\psi_{nm}^{(2)}(x) = (x^2-1)^{m/2}\left(x^2+s^2\right)^{(2+\sqrt{5})/4}\eta\left(x^2,\frac{1}{2},-m,-\frac{\sqrt{5}}{2},0,\lambda;-s^2\right), \tag{20}$$

where $x = \cos\theta$.

Analytic Continuation

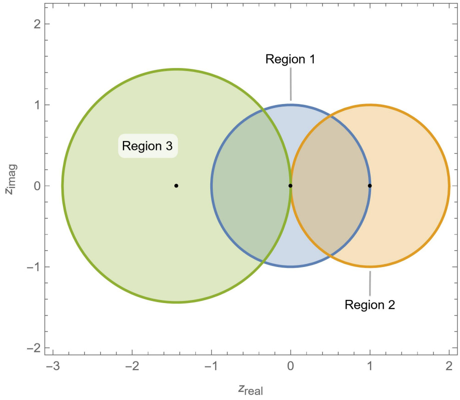

Figure: Regions of convergence of Schäfke η functions. The series centered at $z = 0$ — the one we study in this work — converges inside region 1 (blue); the series centered at $z = 1$ converges inside region 2 (yellow); and, the series centered at $z = -s^2$ converges inside region 3 (green). Our physical domain runs from $|z| \leq 1$. In this figure we consider $s = 1.2$ and the three black dots are located at $z = 0, 1$ and $z = -s^2$.

Eigenfunctions

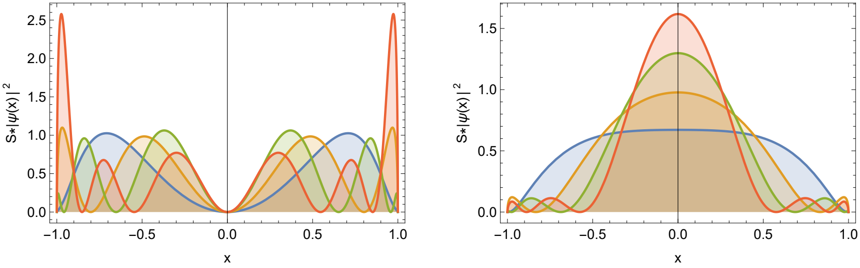

Figure: Plots of the normalized probability densities $S|\psi(x)|^2$, where $x = \cos\theta$. In the left box we plot the first solution (19) for the first four eigenenergies and $m = 0$. In the right box we plot the second solution (20) also for the first four eigenenergies and $m = 0$.

An Analogue Model

Carrying out a $F-$homotopic transformation in equation (11),

$$w(z) = z^{1/2}(z-1)^{m/2}W(z), \tag{21}$$

we need an external potential with the form

$$V_{ext}(z) = \frac{8}{a_0^2}\Big[\frac{U_0(z-1)}{(z+s^2)^3} + \frac{U_1 z}{(z-1)\left(z+s^2\right)^2} + \frac{U_2 z}{\left(z+s^2\right)^3} + \frac{U_3 z^2}{(z-1)^2(z+s^2)} + \frac{U_4}{(z+s^2)}\Big]. \tag{22}$$

Heun's Differential Equation

The resulting equation will be of the form

$$\frac{d^2W}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a_H}\right)\frac{dW}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a_H)}W(z) = 0, \qquad (23)$$

which is the general Heun equation.

Conclusions

- Eigenenergies and eigenfunctions of the black hole problem using Heun's functions;
- Eigenenergies and eigenfunctions of the spheroidal surface problem using Schäfke's η-functions;
- Applying an external field, we created an analogue model for the dynamics of a particle in the background of a Kerr-de Sitter black hole;

Thank you!
Obrigado!

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