Born-Infeld and Charged Black Holes with non-linear source in $f(T)$ Gravity

Manuel Eleuterio Rodrigues

Faculdade de Ciências Exatas e Tecnologia
Universidade Federal do Pará - Campus Universitário de Abaetetuba
IV Amazonian Workshop on Black Holes and Analogue Models of Gravity

May 12, 2015

Ednaldo L. B. Junior, Manuel E. Rodrigues and Mahouton J. S. Houndjo, Born-Infeld and Charged Black Holes with non-linear source in $f(T)$ Gravity in arXiv:1503.07427 [gr-qc].
This paper has been accepted for publication in Journal of Cosmology and Astroparticle Physics.

Abstract

We investigate $f(T)$ theory coupled with a nonlinear source of electrodynamics, for a spherically symmetric and static spacetime in 4D. We re-obtain the Born-Infeld and Reissner-Nordstrom-AdS solutions. We generalize the no-go theorem for any content that obeys the relationship $\mathcal{T}_0{}^0 = \mathcal{T}_1{}^1$ for the energy-momentum tensor and a given set of tetrads. Our results show new classes of solutions where the metric functions are related through $b(r) = Na(r)$. We do the introductory analysis showing that solutions are that of asymptotically flat black holes, with a singularity at the origin of the radial coordinate, covered by a single event horizon. We also reconstruct the action for this class of solutions and obtain the functional form $f(T) = f_0(T)^{(N+3)/[2(N+1)]}$ and $\mathcal{L}_{NED} = \mathcal{L}_0(F)^{(N+3)/[2(N+1)]}$. Using the Lagrangian density of Born-Infeld, we obtain a new class of charged black holes where the action reads $f(T) = -16\beta_{BI}\left[1 + \sqrt{1 + (T/4\beta_{BI})}\right]$.

Outline

1. Introduction to $f(T)$ Gravity;
2. Born-Infeld type black hole solutions;
3. No-Go theorem in $f(T)$ Gravity;
4. New Born-Infeld type solutions in $f(T)$ Gravity;
5. Conclusion.

Outline

1. Introduction to $f(T)$ Gravity;
2. Born-Infeld type black hole solutions;
3. No-Go theorem in $f(T)$ Gravity;
4. New Born-Infeld type solutions in $f(T)$ Gravity;
5. Conclusion.

Outline

1. Introduction to $f(T)$ Gravity;
2. Born-Infeld type black hole solutions;
3. No-Go theorem in $f(T)$ Gravity;
4. New Born-Infeld type solutions in $f(T)$ Gravity;
5. Conclusion.

Outline

1. Introduction to $f(T)$ Gravity;
2. Born-Infeld type black hole solutions;
3. No-Go theorem in $f(T)$ Gravity;
4. New Born-Infeld type solutions in $f(T)$ Gravity;
5. Conclusion.

Outline

1. Introduction to $f(T)$ Gravity;
2. Born-Infeld type black hole solutions;
3. No-Go theorem in $f(T)$ Gravity;
4. New Born-Infeld type solutions in $f(T)$ Gravity;
5. Conclusion.

f(T) Gravity

If we want to describe a spacetime in a coordinate basis, we need a metric $g_{\mu\nu}$ and a connection $\Gamma^{\lambda}{}_{\mu\nu}$. The connection does not have to be related to the metric in general. With this structure, the line element in our manifold, it can be represented local and general basis as

$$dS^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = \eta_{ab}e^{a}e^{b} \ , \ e^{a} = e^{a}{}_{\mu}dx^{\mu}, dx^{\mu} = e_{a}{}^{\mu}e^{a} \ . \quad (1)$$

A general connection cannot be described just in terms of the tetrad (in the same way that the Christoffel symbols are not generically related to the metric components). The following relations hold

$$\Gamma^{\lambda}{}_{\mu\nu} = e_b{}^{\lambda}\partial_\nu e^b{}_\mu + e_a{}^{\lambda}A^a{}_{b\nu}e^b{}_\mu = e_b^{\lambda}D_\nu e^b{}_\mu\,, \qquad (2)$$

where $A^a{}_{b\nu}$ is the spin connection, D_ν is the Lorentz covariant derivative. From (2) we obtain

$$A^a{}_{b\nu} = e^a{}_\lambda\partial_\nu e_b{}^{\lambda} + e^a{}_\lambda\Gamma^{\lambda}{}_{\mu\nu}e_b{}^{\mu} = e^a{}_\lambda\nabla_\nu e_b{}^{\lambda} \qquad (3)$$

where ∇_ν is the connection associated with $\Gamma^{\lambda}{}_{\mu\nu}$.). The torsion tensor is defined by

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}\,. \qquad (4)$$

If we denote the Levi-Civita connection by

$$\bar{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\,, \qquad (5)$$

then we definite the contorsion tensor by

$$K^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \bar{\Gamma}^{\lambda}{}_{\mu\nu} \,. \tag{6}$$

If we impose the metric compatible condition, i.e. $\nabla_\lambda g_{\mu\nu} \equiv 0$, we have

$$K^{\mu\nu}{}_{\alpha} = -\frac{1}{2}\left(T^{\mu\nu}{}_{\alpha} - T^{\nu\mu}{}_{\alpha} - T_{\alpha}{}^{\mu\nu}\right) \,. \tag{7}$$

For facilitating the description of the Lagrangian density and the equations of motion, we can define another tensor from the components of the torsion and contorsion tensors, as

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha} T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha} T^{\beta\mu}{}_{\beta}\right) \,. \tag{8}$$

We can then define the scalar of this theory, the torsion scalar, as follows

$$T = T^{\alpha}{}_{\mu\nu} S_{\alpha}{}^{\mu\nu} \,. \tag{9}$$

Using the definitions listed above, it is a straightforward exercise to show that

$$R^{\rho}{}_{\mu\lambda\nu} = \bar{R}^{\rho}{}_{\mu\lambda\nu} + \bar{\nabla}_{\lambda}K^{\rho}{}_{\mu\nu} - \bar{\nabla}_{\nu}K^{\rho}{}_{\mu\lambda} + K^{\rho}{}_{\sigma\lambda}K^{\sigma}{}_{\mu\nu} - K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\lambda}, \quad (10)$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_{\rho}K^{\rho}{}_{\mu\nu} - \bar{\nabla}_{\nu}K^{\rho}{}_{\mu\rho} + K^{\rho}{}_{\sigma\rho}K^{\sigma}{}_{\mu\nu} - K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\mu\rho}\,, \quad (11)$$

$$R = \bar{R} + T + 2\bar{\nabla}^{\mu}T^{\nu}{}_{\mu\nu}\,. \quad (12)$$

The main requirement of teleparallelism is that there exist a class of frames where the spin connection vanishes, i.e. where $A^{a}{}_{b\nu} \equiv 0$, then we have following Weitzenbok connection

$$\Gamma^{\sigma}{}_{\mu\nu} = e_{a}{}^{\sigma}\partial_{\nu}e^{a}{}_{\mu} = -e^{a}{}_{\mu}\partial_{\nu}e_{a}{}^{\sigma}\,. \quad (13)$$

With this connection, the Riemann tensor is identically null, $R^{\alpha}{}_{\beta\mu\nu} \equiv 0$. Then we have $R \equiv 0$, or

$$T = -\bar{R} - 2\bar{\nabla}^{\mu}T^{\nu}{}_{\mu\nu}\,. \quad (14)$$

First we take the following Lagrangian density

$$\mathcal{L} = e\left[f(T) + 2\kappa^2\mathcal{L}_{NED}(F)\right] , \tag{15}$$

where $\mathcal{L}_{NED}(F)$ is the contribution of non-linear electrodynamics (NED), with $F = (1/4)F_{\mu\nu}F^{\mu\nu}$ being $F_{\mu\nu}$ the components of the Maxwell's tensor, $e = \det[e^a{}_\mu] = \sqrt{-g}$, $g = \det[g_{\mu\nu}]$, and $\kappa^2 = 8\pi G/c^4$, where G is the Newtonian constant and c the speed of light. To establish the equations of motion by Euler-Lagrange ones, we have to take the derivations with respect to the tetrads, then one gets the following equations of motion

$$S_\beta{}^{\mu\alpha}\partial_\alpha T f_{TT}(T) + \left[e^{-1}e^a{}_\beta\partial_\alpha\left(ee_a{}^\sigma S_\sigma{}^{\mu\alpha}\right) + T^\sigma{}_{\nu\beta}S_\sigma{}^{\mu\nu}\right]f_T(T)$$
$$+\frac{1}{4}\delta^\mu_\beta f(T) = \frac{\kappa^2}{2}\mathcal{T}_\beta{}^\mu \tag{16}$$

where $\mathcal{T}_{\beta}{}^{\mu}$ is the energy-momentum tensor of the source of non-linear electrodynamics

$$\mathcal{T}_{\beta}{}^{\mu} = -\frac{2}{\kappa^2}\left[\delta^{\mu}_{\beta}\mathcal{L}_{NED}(F) - \frac{\partial \mathcal{L}_{NED}(F)}{\partial F} F_{\beta\sigma}F^{\mu\sigma}\right] . \qquad (17)$$

We consider a spherically symmetric and static spacetime. The set of chosen tetrads is given by

$$[e^{a}{}_{\mu}] = \begin{bmatrix} e^{a(r)/2} & 0 & 0 & 0 \\ 0 & e^{b(r)/2}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{b(r)/2}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{b(r)/2}\cos\theta & -r\sin\theta & 0 \end{bmatrix} \qquad (18$$

which generates the metric

$$dS^2 = e^{a(r)}dt^2 - e^{b(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (19)$$

The torsion scalar is given by

$$T = \frac{2}{r}\left[-e^{-b/2}\left(a' + \frac{2}{r}\right) + e^{-b}\left(a' + \frac{1}{r}\right) + \frac{1}{r}\right] . \qquad (20)$$

By spherical symmetry and staticity of the metric, considering only electric charge, it can be shown that the Maxwell field has only a single component $F^{10}(r)$. Making use of Euler-Lagrange equations for the field A_μ (the for-potential), we obtain the modified Maxwell equation

$$\bar{\nabla}_\mu\left[F^{\mu\nu}\frac{\partial \mathcal{L}_{NED}}{\partial F}\right] = 0 \, , \qquad (21)$$

whose solution, using $\nu = 0$, is given by

$$F^{10}(r) = \frac{q}{r^2}e^{-(a+b)/2}\left(\frac{\partial \mathcal{L}_{NED}}{\partial F}\right)^{-1} . \qquad (22)$$

Finally, the equations of motion for this theory are given by

$$2\frac{e^{-b}}{r}\left(e^{b/2}-1\right)T'f_{TT}+\frac{e^{-b}}{r^2}\left[b'r+\left(e^{b/2}-1\right)(a'r+2)\right]f_T$$
$$+\frac{f}{2}=-2\mathcal{L}_{NED}-2\frac{q^2}{r^4}\left(\frac{\partial\mathcal{L}_{NED}}{\partial F}\right)^{-1}, \tag{23}$$

$$\frac{e^{-b}}{r^2}\left[\left(e^{b/2}-2\right)a'r+2\left(e^{b/2}-1\right)\right]f_T+\frac{f}{2}=-2\mathcal{L}_{NED}$$
$$-2\frac{q^2}{r^4}\left(\frac{\partial\mathcal{L}_{NED}}{\partial F}\right)^{-1}, \tag{24}$$

$$\frac{e^{-b}}{2r}\left[a'r+2\left(1-e^{b/2}\right)\right]T'f_{TT}+\frac{e^{-b}}{4r^2}\left[\left(a'b'-2a''-a'^2\right)r^2+\right.$$
$$\left.+\left(2b'+4a'e^{b/2}-6a'\right)r-4e^b+8e^{b/2}-4\right]f_T+\frac{f}{2}=-2\mathcal{L}_{NED} \tag{25}$$

A bit of History

1- M. Born and L. Infeld, *Foundations of the New Field Theory*, Proc. R. Soc. (London) A **144**, 425-451 (1934), [DOI: 10.1098/rspa.1934.0059];
2- B. Hoffmann and L. Infeld, *On the Choice of the Action Function in the New Field Theory*, Phys. Rev. **51** 765-773 (1937), [DOI: 10.1103/PhysRev.51.765];
3- A. Peres, *Nonlinear Electrodynamics in General Relativity*, Phys. Rev. **122** 273-274 (1961), [DOI: 10.1103/PhysRev.122.273];
4- J. Bardeen, presented at GR5, Tiflis, U.S.S.R., and published in the conference proceedings in the U.S.S.R. (1968);
5- R. Pellicer and R.J. Torrence, *Nonlinear electrodynamics and general relativity*, J.Math.Phys. **10** (1969) 1718-1723, [DOI: 10.1063/1.1665019].

Here we will show that $f(T)$ Gravity with NED source leads to all the known solutions of the GR coupled to nonlinear electrodynamics. Let us perform here some best known cases, starting with the Lagrangian density of Born-Infeld (BI). The Lagrangian density is given by BI

$$\mathcal{L}_{NED} = \mathcal{L}_{BI} = 4\beta_{BI}^2 \left[1 - \sqrt{1 + \frac{F}{2\beta_{BI}^2}} \right] , \tag{26}$$

with $F = (1/4)F_{\mu\nu}F^{\mu\nu}$ and β_{BI} being the BI parameter. This Lagrangian density falls into the case of Maxwell electromagnetic theory, for the limit where the BI parameter tends to infinity, then

$$\lim_{\beta_{BI}\to\infty} \mathcal{L}_{BI} \sim -F + O\left[F^2\right] \sim \mathcal{L}_{Maxwell} . \tag{27}$$

Let us look at the particular cas of the TT, where $f(T) = T - 2\Lambda$, $f_T(T) = 1$ and $f_{TT}(T) = 0$. In this case we reobtain the BI solution

$$e^{a(r)} = e^{-b(r)} = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + 4\beta_{BI}\Big(2\frac{\beta_{BI}}{3}r^2 - \frac{1}{r}\int_{\infty}^{r}\sqrt{q^2 + 4\beta_{BI}^2 r^4}dr\Big)\,, F^{10}(r) = \frac{2\beta_{BI}q}{\sqrt{q^2 + 4\beta_{BI}^2 r^4}}\,. \quad (28)$$

Taking now the particular case where β_{BI} goes to infinity, the Lagrangian density $\mathcal{L}_{NED} = \mathcal{L}_{Maxwell} = -F$ and $\partial\mathcal{L}_{NED}/\partial F = -1$.

We will show now that the solutions of type $b(r) = -a(r)$ can exist only in the particular case of the TT. To do this, we will subtract the equation (24) from (23), yielding after rearrangement

$$\frac{d}{dr}\left[\ln f_T(T)\right] + \left[\frac{a' + b'}{2\left(e^{b/2} - 1\right)}\right] = 0\,. \tag{29}$$

In this case, we can enunciate the theorem as follows:
Theorem: The solutions of a Weitzenbock spacetime with spherical and static symmetry whose the sets of tetrads are given (18), with the restriction $b(r) = -a(r)$ and $\mathcal{T}_0{}^0 = \mathcal{T}_1{}^1$ in (16), exist if and only if the theory is Teleparallel, where $f(T)$ is a linear function of the torsion scalar.

For $b(r) = -2a(r)$, we have the following solution

$$dS^2 = \left(1 - \frac{2r_0}{r^2}\right) dt^2 - \left(1 - \frac{2r_0}{r^2}\right)^{-2} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (30)$$

$$F^{10}(r) = -\frac{4r_0^2}{qr^3}\sqrt{1 - \frac{2r_0}{r^2}}, \quad (31)$$

$$T(r) = -\frac{8r_0^2}{r^6}, f(T) = -\frac{6}{5}\sqrt{2}r_0^{1/3}(-T)^{5/6}, \quad (32)$$

$$\mathcal{L}_{NED} = \frac{3q^{5/3}}{5\sqrt{2}r_0^4}(-F)^{5/6}, R(r) = \frac{4r_0}{r^6}(r^2 - 2r_0). \quad (33)$$

Important observations. We can now note a certain logic for the solutions $b = -2a, -4a, -6a$. Let us generalize the reconstructions this cases for

$$f(T) = f_0 (-T)^{(N+3)/[2(N+1)]}, \ \mathcal{L}_{NED} = \mathcal{L}_0 (-F)^{(N+3)/[2(N+1)]}, \quad (34)$$

with an even N. Only a particular case is admitted here, when $N = 1$ we recover the TT with the Maxwell Lagrangian density. All of these solutions are asymptotically Minkowskian. The effect of the electric field vanishes at spatial infinity. The integration constant r_0 appearing in all solutions should be that depend implicitly on the mass and charge of the black hole, so that if we take the limit $\lim_{m,q\to 0} r_0 \to 0$, thus returning to the Minkowskian vacuum.

When we choose an odd N, we have very complicated differential equations where the analytical solutions are almost impossible to be obtained. Thus, we do not take them into account.

We can conclude that the cases with $N = 3, 5, 7, ...$ lead to more complicated differential equations that do not yield atypical solutions. However with $N = 4, 6, ...$ there is no event horizon, being a non interesting geometries.
Now, choosing the Lagrangian $\mathcal{L}_{NED} \equiv \mathcal{L}_{BI}$, we can build the following solution

$$dS^2 = \exp\left[2\int \frac{dr}{\frac{\sqrt{q^2+4\beta_{BI}^2 r^4}}{q\sqrt{2\beta_{BI}}} - r}\right] dt^2 - \frac{2(q^2+4\beta_{BI}^2 r^4)}{\left(\sqrt{2q^2+8\beta_{BI}^2 r^4} - 2qr\sqrt{\beta_{BI}}\right)^2} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (35)$$

(36)

$$F^{10}(r) = \frac{2q\beta_{BI}}{q^2 + 4\beta_{BI}^2 r^4}\left(qr\sqrt{2\beta_{BI}} - \sqrt{q^2 + 4\beta_{BI}^2 r^4}\right)$$

$$\exp\left[-\int \frac{dr}{\frac{\sqrt{q^2+4\beta_{BI}^2 r^4}}{q\sqrt{2\beta_{BI}}} - r}\right], \tag{37}$$

$$T(r) = -\frac{4q^2\beta_{BI}}{q^2 + 4\beta_{BI}^2 r^4}, f(T) = -16\beta_{BI}^2\left[1 - \sqrt{1 + \frac{T}{4\beta_{BI}}}\right] \tag{38}$$

We have a solution of a charged black hole, spherically symmetric and static, and asymptotically Minkowskian, because of $e^{a(r)}, e^{b(r)} \to 1$ for $r \to +\infty$. The expression of the curvature scalar is too large to be written here, but in the limit of infinite space, it vanishes, and for the limit of the radial coordinate origin, it diverges, indicating a singularity. On the other hand $F^{10}(r)$ goes to zero at spatial infinity and is regular, with the value $-2\beta_{BI}$, at the origin $r = 0$. The solution possesses two event horizons in

$$r_{\pm} = \frac{q}{2\sqrt{\beta_{BI}}}\sqrt{1 \pm \frac{1}{q}\sqrt{q^2 - 4}}\,. \tag{39}$$

The extreme limit is given for $q = 2$, where $r_+ = r_- = 1/\sqrt{\beta_{BI}}$.

Conclusion

Here we reconsidered the original idea of BI for a nonlinear Lagrangian density in F implying that for certain relationship between the metric functions $a(r)$ and $b(r)$, a function $f(T)$ with the same functional dependence of the NED action. We can re-obtain so the famous BI solutions with cosmological constant and the one of Reissner-Nordstrom-AdS, of GR.
A very strong result is established by no-go theorem where we only can have solutions in the $f(T)$ theory with non-linear terms in T, when we break down the famous symmetry $b(r) = -a(r)$ and $\mathcal{T}_0{}^0 = \mathcal{T}_1{}^1$ for the tetrads (18).

We performed a powerful algorithm for solving differential equations of this theory. We see that still can be addressed more new cases in which the relationship between the metric functions are different from those used here. It is also possible to consider metric functions such that the solution is a regular black hole in $f(T)$ Gravity. This is currently being done by us in another work. The new solutions that merit to be pointed out are charged and asymptotically flat black holes solutions whose Lagrangian densities and action functions are given by $\mathcal{L}_{NED} = \mathcal{L}_0\left(-F\right)^{(N+3)/[2(N+1)]}$ and $f(T) = f_0\left(-T\right)^{(N+3)/[2(N+1)]}$, when $b(r) = -Na(r)$ with an even N.

Another new solution obtained here is when we consider the no-go theorem and the Lagrangian density BI. For this model the solution is asymptotically flat with two or one horizon (extreme case). Interestingly, the functional form of $f(T)$ is identical to the Lagrangian density of BI, but with $F \to T$, which recovers the original idea of Ferraro and Fiorini, but concerning black hole here.

MUITO OBRIGADO

Another new solution obtained here is when we consider the no-go theorem and the Lagrangian density BI. For this model the solution is asymptotically flat with two or one horizon (extreme case). Interestingly, the functional form of $f(T)$ is identical to the Lagrangian density of BI, but with $F \to T$, which recovers the original idea of Ferraro and Fiorini, but concerning black hole here.

MUITO OBRIGADO