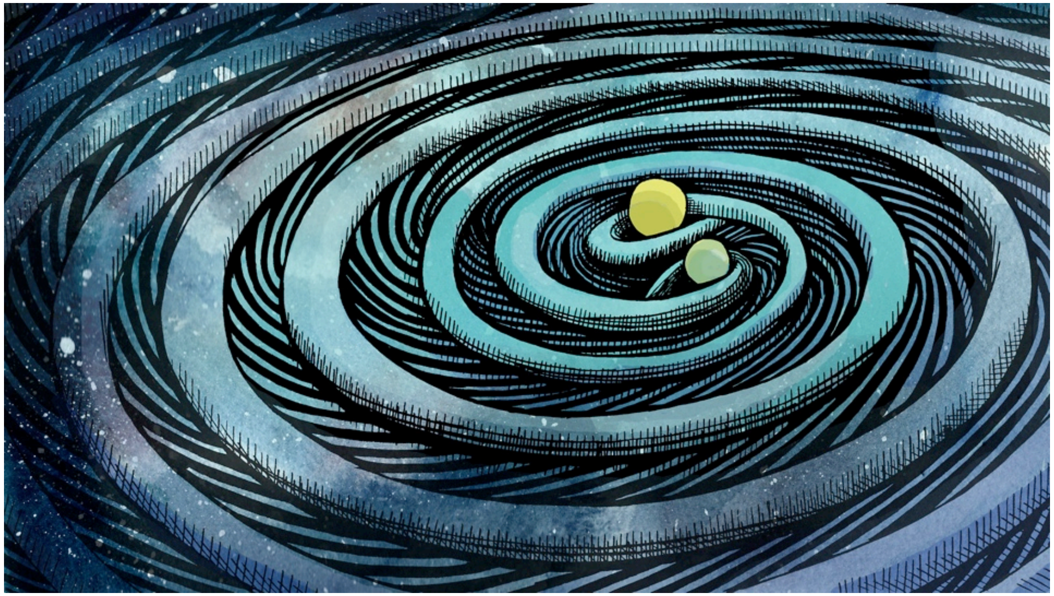
Numerical-relativity modelling of astrophysical sources of gravitational waves



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Outline

Lecture 1: Hydrodynamics and MHD

Lecture 2: Einstein's equations and NR

Lecture 3: Numerical methods

Lecture 4: Applications in astrophysics

- Binary neutron star mergers
- Core collapse supernovae

Lecture 2
Einstein's equations and Numerical Relativity

Einstein's equations and Numerical Relativity

The dynamics of the gravitational field is described by Einstein's field equations:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

These equations relate the **spacetime geometry** (left-hand side) with the **distribution of matter and energy** (right-hand side): *"Matter tells spacetime how to curve, and spacetime tells matter how to move."*

Einstein's equations are a system of 10 nonlinear, coupled, partial differential equations in 4 dimensions.

When written with respect to a general coordinate system they may contain hundreds of terms ...

There's plenty of **exact solutions** of Einstein's equations, but very **few of such solutions have astrophysical significance.** Due to their complexity exact solutions of such equations have only been found when adopting simplifying symmetries:

- **Schwarzschild solution** (static and spherically symmetric)
- **Kerr solution** (stationary and axisymmetric)
- **Cosmological solution** (isotropic, homogeneous, or both)

When studying more complex systems with astrophysical significance (gravitational collapse, mergers of compact binaries) unfeasible to solve Einstein's equations in an exact way.

Numerical Relativity emerged in the mid 1960s from the need to study such kind of problems, aiming at trying to solve the field equations with supercomputers using numerical approximations.

NR's current main **goal**: provide templates of the gravitational produced in astrophysical sources to assist detection with current and next generation detectors (LIGO/VIRGO/KAGRA/ET/CE).

Procedure to derive the 3+1 equations

1. Foliation of the 4-dim spacetime with 3-dim spatial hypersurfaces defined by a scalar function, the temporal coordinate. This geometrical construction defines a unit normal vector to the hypersurfaces.

2. Split of 4-dim spacetime tensors into their temporal and spatial parts, using the normal vector and the spatial metric.

3. Re-writing of Einstein's equations using such split tensors.

4. Choice of a natural direction for the time evolution.

5. Choice of a coordinate basis to express all equations.

Einstein's equations in 3+1 form

Using the projection operator and the normal vector, Einstein's equations can be separated in **three groups:**

- Normal projection (1 equation; energy or Hamiltonian constraint)

$$n^\alpha n^\beta (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

- Mixed projections (3 equations; momentum constraints)

$$P[n^\alpha (G_{\alpha\beta} - 8\pi T_{\alpha\beta})] = 0$$

- Projection onto the hypersurface (6 equations; evolution of the extrinsic curvature)

$$P(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

First step: Spacetime foliation

Spacetime: differentiable manifold $\mathcal{M}$ with a Lorentzian metric $g_{\mu\nu}$ of signature +2, $(\mathcal{M}, g)$

Such manifold is covered by a coordinate chart $\{x^\mu\}$ with $\mu = 0, \cdots, 3$

A coordinate basis of the tangent space of $\mathcal{M}$ en $p, T_p\mathcal{M}$, is given by $\partial_\mu \equiv \partial/\partial x^\mu$

A vector $V \in T_p\mathcal{M}$ is expressed as $V = V^\mu \partial_\mu$

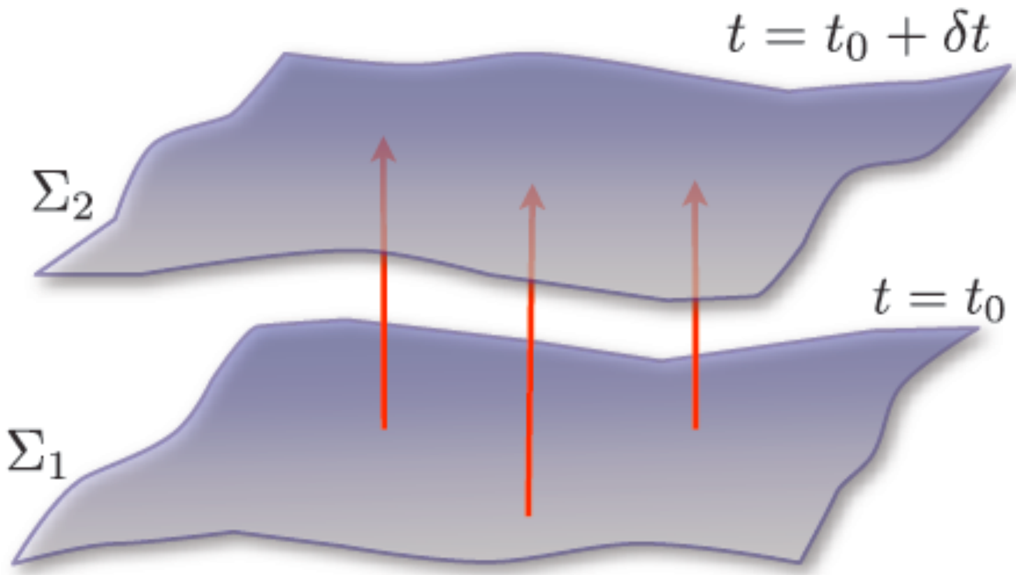
where V^μ are the components of V on the basis ∂_μ

A 1-form of the cotangent space $\Omega \in T_p^*\mathcal{M}$ is an object that is dual to a vector, that is, it leads to a number when acting on a vector. The simplest example is the differential of a function

$$df = \partial_\mu f \, dx^\mu \qquad \{dx^\mu\} \text{ natural basis in } T_p^*\mathcal{M}$$

Therefore, an arbitrary 1-form can be written as $\Omega = \Omega_\mu dx^\mu$

Foliation of spacetime with three-dimensional spatial hypersurfaces defined by a scalar function. Such function is the temporal coordinate t, as we shall see.



We define therefore the 1-form $\Omega_\mu \equiv \nabla_\mu t$

such that $|\Omega|^2 = g^{\mu\nu}\nabla_\mu t \nabla_\nu t \equiv -\alpha^{-2}$

this defines the lapse function that is strictly positive for spatial hypersurfaces $\alpha(t, x^i) > 0$

The lapse function allows to do two important things:

1. define the normal unit vector to the hypersurface Σ

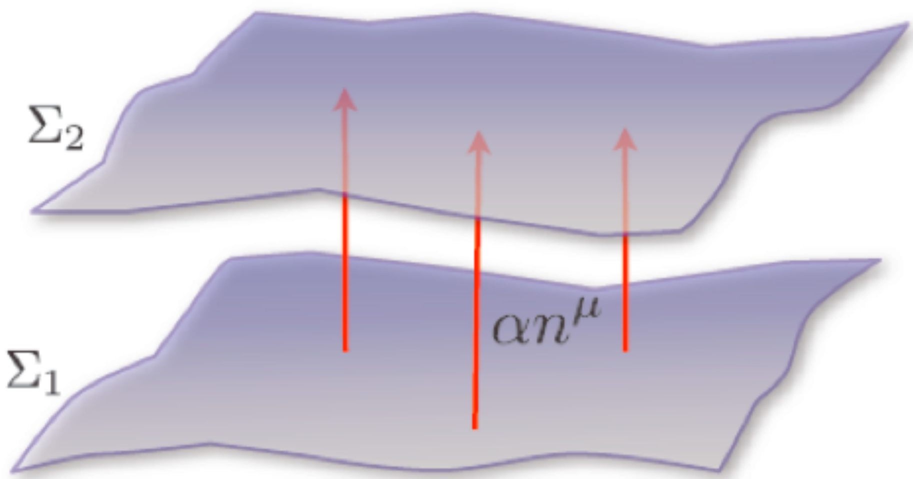
$$n^\mu \equiv -\alpha g^{\mu\nu}\Omega_\nu = -\alpha g^{\mu\nu}\nabla_\nu t$$

with $n^\mu n_\mu = -1$ (quiz: prove it!)

the negative sign is chosen so that the normal vector points in the direction of increasing *t*

2. define the spatial metric (induced)

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$



Second step: split 4-dim tensors

The normal vector and the spatial metric are two useful tools to split any 4-tensor in a purely spatial (on the hypersurface) and a purely temporal part (orthogonal to Σ and along $\mathbf{n}$).

The spatial part of a tensor is obtained by contraction with the spatial projector operator:

$$\gamma^{\mu}_{\ \nu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu}_{\ \nu} + n^{\mu}n_{\nu} = \delta^{\mu}_{\ \nu} + n^{\mu}n_{\nu}$$

while the temporal part is obtained by contraction with the temporal projector operator:

$$N^{\mu}_{\ \nu} = -n^{\mu}n_{\nu}$$

(quiz: prove it!)

with the two projectors being obviously orthogonal: $\gamma^{\nu}_{\ \mu}N^{\mu}_{\ \nu} = 0$

This allows to define the covariant 3-derivative of a spatial tensor. It is simply the projection on to the hypersurface of all the indices of the covariant 4-derivative

$$D_\alpha T^\beta{}_\delta = \gamma^\rho{}_\alpha \gamma^\beta{}_\sigma \gamma^\tau{}_\delta \nabla_\rho T^\sigma{}_\tau$$

compatible with the spatial metric: $D_\alpha \gamma^\beta{}_\delta = 0$

All the algebra of 4-tensors can thus be immediately applied on the spatial hypersurface, so that the covariant 3-derivative can be expressed in terms of the 3-dim connection coefficients (Christoffel symbols)

$$\Gamma^\alpha{}_{\beta\delta} = \frac{1}{2}\gamma^{\alpha\mu}(\gamma_{\mu\beta,\delta} + \gamma_{\mu\delta,\beta} - \gamma_{\beta\delta,\mu})$$

Notation: $\gamma_{\mu\beta,\delta} \equiv \partial_\delta \gamma_{\mu\beta}$

Likewise, the 3-dim Riemann tensor (curvature) associated with γ is defined through the double covariant 3-derivative of any spatial vector $\mathbf{W}$, that is

$$2D_{[\alpha}D_{\beta]}W_\delta = R^\mu{}_{\delta\alpha\beta}W_\mu \quad \text{(Ricci identity)}$$

where $R^\mu{}_{\delta\alpha\beta}n_\mu = 0$ $\qquad 2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$

Remember that, explicitely, the 3-dim Riemann tensor can be expressed in terms of the 3-dim connection coefficients:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\mu{}_{\beta\delta}\Gamma^\alpha{}_{\mu\gamma} - \Gamma^\mu{}_{\beta\gamma}\Gamma^\alpha{}_{\mu\delta}$$

Moreover, the contractions of the 3-dim Riemann tensor, that is, the Ricci tensor and the Ricci scalar, are given by:

$$R_{\alpha\beta} = R^\delta{}_{\alpha\delta\beta} \quad \text{and} \quad R = R^\delta_\delta$$

Important not to confuse the 3-dim Riemann tensor $R^{\mu}{}_{\delta\alpha\beta}$ with the corresponding 4-dim tensor ${}^{(4)}R^{\mu}{}_{\delta\alpha\beta}$

$R^{\mu}{}_{\delta\alpha\beta}$ is purely spatial (spatial derivatives of the spatial metric γ)

${}^{(4)}R^{\mu}{}_{\delta\alpha\beta}$ is a 4-dim object that also contains temporal derivatives of the 4-metric g

The information present in ${}^{(4)}R^{\mu}{}_{\delta\alpha\beta}$ and absent in $R^{\mu}{}_{\delta\alpha\beta}$ can be found in another spatial tensor, the extrinsic curvature.

Intrinsic and extrinsic curvature of spatial hypersurfaces:

Intrinsic curvature given by the 3-dimensional Riemann tensor defined in terms of the 3-metric γ_{ij} .

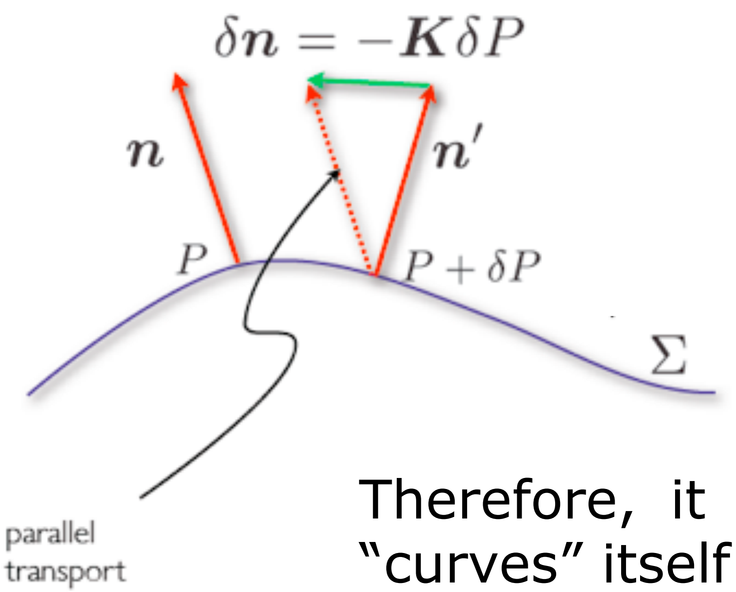
Extrinsic curvature K_{ij} measures the change of the vector normal to the hypersurface as it is parallel-transported from one point in the hypersurface to another.

A common definition of the extrinsic curvature is in terms of the Lie derivative:

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n \gamma_{\alpha\beta}$$

The Lie derivative can be seen as a generalization of the directional derivative. For a generic tensor of rank $\binom{1}{1}$ it is given by:

$$\mathcal{L}_{\boldsymbol{X}} T^{\mu}{}_{\nu} = X^{\alpha} D_{\alpha} T^{\mu}{}_{\nu} - T^{\alpha}{}_{\nu} D_{\alpha} X^{\mu} + T^{\mu}{}_{\alpha} D_{\nu} X^{\alpha}$$



Geometrically the extrinsic curvature measures the change of the normal vector to the hypersurface as this vector is parallel-transported from one point of the hypersurface to another.

Therefore, it measures how the 3-dim hypersurface "curves" itself w.r.t. to the 4-dim spacetime.

Third step: split of Einstein's equations

Next, we need to split Einstein's equations in their spatial and temporal parts.

$$ {}^{(4)}G_{\mu\nu} \equiv {}^{(4)}R_{\mu\nu} - \frac{1}{2}\,{}^{(4)}R\, g_{\mu\nu} = 8\pi T_{\mu\nu} $$

To do so, we need to use a few identities:

1. Gauss-Codazzi equations: split of the 4-dim Riemann tensor projecting all of its indices on to the hypersurface

$$ R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\delta}\gamma^{\sigma}{}_{\gamma}\,{}^{(4)}R_{\mu\nu\rho\sigma} $$

2. Codazzi-Mainardi equations: three spatial and one temporal projections of the 4-dim Riemann tensor

$$ D_{\alpha}K_{\beta\gamma} - D_{\beta}K_{\alpha\gamma} = \gamma^{\rho}{}_{\beta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\gamma}n^{\sigma}\,{}^{(4)}R_{\rho\mu\nu\sigma} $$

Dem.- See Wald, *General Relativity*, University of Chicago Press (1984)

We also need to analyse the projections of the stress-energy tensor, as they are relevant when considering the r.h.s. of Einstein's equations.

For a perfect fluid, the stress-energy tensor is given by:

$$T_{\mu\nu} = (e+p)u_\mu u_\nu + p\, g_{\mu\nu} = \rho h u_\mu u_\nu + p\, g_{\mu\nu}$$

where e, p, h and ρ are, respectively the energy density, the pressure, the specific enthalpy and the rest-mass density of the fluid, $\rho h = e + p$

As $n^\mu u_\mu = 1$ (the two vectors are parallel and unitary) the density of energy measure by the normal observers is given by the doble temporal projection

$$e = n^\mu n^\nu T_{\mu\nu} \qquad \text{(quiz: prove it!)}$$

Likewise, the momentum density (the mass current) is given by the mixed spatial and temporal projections

$$j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} = -(e+p)(u_\mu + n_\mu) \qquad \text{(quiz: prove it!)}$$

Hamiltonian constraint equation

First, we take a double temporal projection of the l.h.s. of Einstein's equations, to obtain

$$2n^\mu n^\nu \, {}^{(4)}G_{\mu\nu} = R + K^2 - K_{\mu\nu}K^{\mu\nu}$$

Doing the same for the r.h.s., and using the Gauss-Codazzi equations contracted twice with the spatial metric, and using the definition of the energy density, we obtain (after some algebra) the Hamiltonian constraint equation:

$$\boxed{R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi e}$$

This is an elliptic equation (and therefore contains no temporal derivatives) that must be satisfied everywhere on the spatial hypersurface Σ

Momentum constraint equation

Likewise, with a mixed spatial-temporal projection of the l.h.s. of Einstein's equations we obtain

$$\gamma^{\alpha\mu} n^{\nu} \, {}^{(4)}G_{\mu\nu} = \gamma^{\alpha\mu} n^{\nu} R_{\mu\nu}$$

so that, using the Codazzi-Mainardi in the r.h.s. of this equation, we arrive to

$$\gamma^{\alpha\mu} n^{\nu} \, {}^{(4)}G_{\mu\nu} = D^{\alpha} K - D_{\mu} K^{\alpha\mu}$$

On the other hand, the same contraction in the r.h.s. of Einstein's equations leads to

$$\gamma^{\alpha\mu} n^{\nu} T_{\mu\nu} = -j^{\alpha}$$

Both sides are equal, which leads to the momentum constraint equation:

$$\boxed{D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}}$$

which are three elliptic equations.

Fourth step: choice of direction for time evolution

Evolution equations to describe how $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ evolve in time, form one spatial hypersurface to the next, can be obtained from the definition of the extrinsic curvature and the Ricci equations (two spatial and two temporal projections of the 4-dim Riemann tensor):

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n \gamma_{\alpha\beta}$$

$$\mathcal{L}_n K_{\alpha\beta} = n^\delta n^\gamma \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \, {}^{(4)}R_{\nu\delta\mu\sigma} - \frac{1}{\alpha} D_\alpha D_\beta \alpha - K^\gamma{}_\beta K_{\alpha\gamma}$$

However, the Lie derivative along the unit normal is not the natural time derivative orthogonal to the hypersurfaces.

This is so because n^α is not dual to the 1-form Ω_α of the hypersurface, that is

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha \Omega^\mu \Omega_\mu = \frac{1}{\alpha} \neq 1$$

Therefore, it is necessary to find a new vector along which to carry out the temporal evolution, and dual to the hypersurface 1-form. Such a vector can be easily defined as:

$$t^\mu \equiv \alpha n^\mu + \beta^\mu$$

where β^μ is any "shift" spatial vector.

Clearly these two objects are now dual to each other:

$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = \frac{\alpha}{\alpha} = 1$$

Therefore, the lapse function and the shift vector jointly determine the evolution of the coordinates from one hypersurface to the next.

The lapse function determines the amount of elapsed proper time between two consecutive hypersurfaces along the unit normal, while the shift vector determines the amount by which the spatial coordinates have shifted w.r.t. to the normal vector.

With this definition let us consider again the Lie derivative along the unit normal vector $\mathcal{L}_n$. Since

$$\alpha \mathcal{L}_n = \mathcal{L}_t - \mathcal{L}_\beta$$

the definition of extrinsic curvature

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n \gamma_{\alpha\beta}$$

can be rewritten as

$$\boxed{\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}}$$

This expression shows that the extrinsic curvature is a measure of the rate of change of the spatial metric

$$K_{\mu\nu} \propto -\frac{1}{\alpha}\mathcal{L}_t \gamma_{\mu\nu}$$

The previous equation is a *definition* and not one of Einstein's equations, something that is usually confused.

The evolution part of Einstein's equations

We can now obtain the remaining part of the 3+1 split, that is, the part of Einstein's equations that describes their time evolution.

As for the constraint equations, we need appropriate projections of both sides of Einstein's equations, in particular two spatial projections, that is:

$$\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\,{}^{(4)}G_{\mu\nu} = 8\pi S_{\mu\nu} \equiv 8\pi\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}T_{\mu\nu}$$

Using the Ricci equations and after some algebra, we obtain:

$$\boxed{\begin{aligned} \mathcal{L}_t K_{\mu\nu} = \quad & - \quad D_\mu D_\nu \alpha + \alpha\left({}^{(3)}R_{\mu\nu} - 2K^{\beta}{}_{\nu}K_{\mu\beta} + KK_{\mu\nu}\right) \\ & - \quad 8\pi\alpha\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}(S-e)\right) + \mathcal{L}_\beta K_{\mu\nu} \end{aligned}}$$

where $S \equiv S^{\mu}{}_{\mu}$

Fifth step: choice of a coordinate basis

So far we have dealt with tensor equations and we have not specified a coordinate basis with unit vectors e_j

Doing this allows us to simplify the equations and to highlight the spatial nature of γ and K

In our case the choice is very simple as we want that:

1. three of the vectors must be purely spatial, that is

$$n_\mu (e_j)^\mu = 0 \to (e_1)^\mu = (0,1,0,0) \quad \text{for instance}$$

2. the fourth vector has to point in the direction of vector t

$$(e_0)^\mu = t^\mu = (1,0,0,0)$$

As a result

1. $$\mathcal{L}_t = \partial_t$$

that is, the Lie derivative along t is a simple partial derivative.

2. $$n_j = n_\mu (e_j)^\mu = 0 \quad \text{but} \quad n_0 \neq 0$$

that is, the spatial covariant components of a temporal vector vanish; only the temporal component is not zero.

3. $$n_\mu \beta^\mu = \beta^0 n_0 = 0 \;\Rightarrow\; \beta^0 = 0 \;\Rightarrow\; \beta^\mu = (0, \beta^j)$$

that is, the temporal contravariant component of a spatial vector vanishes; only the spatial components are not zero.

With all of this and keeping in mind that $n_\mu n^\mu = -1$ we obtain

$$\boxed{n^\mu = \frac{1}{\alpha}(1, -\beta^i) \qquad n_\mu = (-\alpha, 0, 0, 0)}$$

(quiz: prove it!)

It can also be shown that the contravariant components of the metric in the 3+1 split are:

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

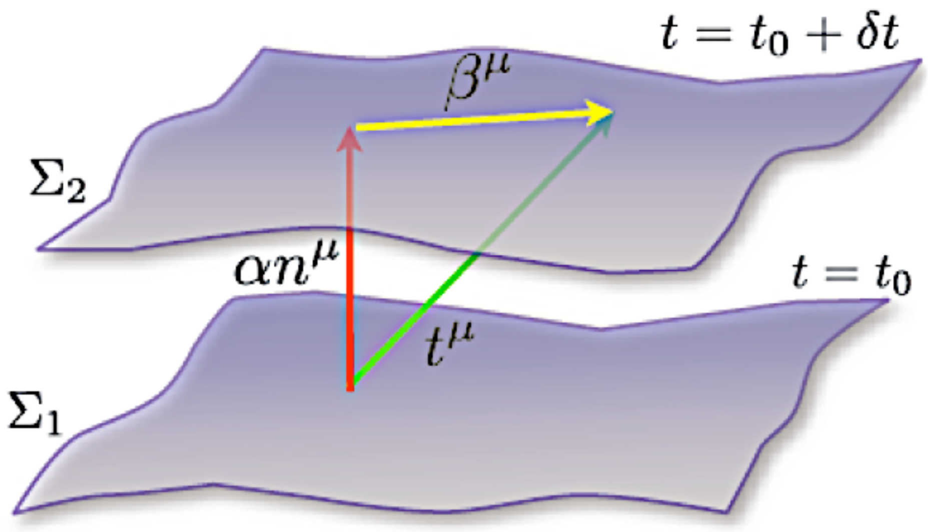
while the covariant components are

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

Note that $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$, i.e. the two matrices are inverse of one another and can be used to raise and lower indices of spatial tensors.

We can now have a more intuitive interpretation of the lapse function, the shift vector, and the spatial metric. Using the expression of the covariant 4-dim metric, we can write the line element as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



Therefore:

the lapse function measures the proper time between two adjacent hipersurfaces

$$d\tau^2 = -\alpha^2(t, x^j)dt^2$$

the shift vector relates the spatial coordinates between two adjacent hypersurfaces

$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)dt$$

the spatial metric measures distances between points on each hypersurface

$$dl^2 = \gamma_{ij}dx^i dx^j$$

Summary: 3+1 ADM system

$$\begin{aligned}\partial_t K_{ij} = & - & D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ & - & 8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_\beta K_{ij})\end{aligned}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

These 6+6+3+1 equations are known as **ADM equations**. In practice only the evolution equations are solved and the constraint equations are used to monitor the quality of the numerical solution.

Cauchy (Initial Value) Problem

The ADM equations constitute a Cauchy problem where the PDEs are solved given certain initial conditions on the initial hypersurface.

- Specify the initial data γ_{ij} , K_{ij} at *t=0* subjected to the constraint equations.

- Specify the coordinates through the (freely specifiable) lapse function α and shift vector β^i

- Evolve the initial data to the next time step using the Einstein's equations and the definition of the extrinsic curvature K_{ij}

Original references:

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Nowadays, the ADM equations are hardly used in Numerical Relativity.

While the ADM equations have no peculiarities from a mathematical point of view, the numerical experience with those equations has shown that they are not suitable for a numerical approach.

In particular, it has been shown that the ADM system is weakly hyperbolic and, therefore, constitutes an ill-posed Cauchy (IVP) problem.

In practice, the ADM system is prone to the appearance of numerical instabilities that destroy the solution (exponentially unstable growing modes).

However, the stability properties of numerical implementations can be improved by introducing new auxiliary functions and rewriting the ADM equations in terms of those new functions.

The concept of hyperbolicity

Let us consider a first-order system of evolution equations:

$$\partial_t u + M^i \partial_i u = s(u)$$

where M^i are $n \times n$ matrices and $i = 1, 2, 3$

Let us consider an arbitrary unit vector n_i and let us build the matrix $P = M^i n_i$ the so-called system's principal symbol.

The system can be classified as:

- Strongly hyperbolic: if P has real eigenvalues and there exists a complete set of eigenvectors for any n_i .

- Weakly hyperbolic: if P has real eigenvalues but there does not exist a complete set of eigenvectors.

- Symmetric hyperbolic: if P is a symmetric matrix regardless of n_i . They are, therefore, strongly hyperbolic.

Only strong and symmetric hyperbolic systems are well-posed. Eigenvalues represent propagation speeds of the system.

ADM vs Maxwell

The ADM equations look rather cryptic and complicated. The analogy between these equations and Maxwell's equations helps to better understand the ADM system.

In electromagnetism, the relevant quantities are the electric and magnetic fields, the charge density, and the charge current density, that is: $\mathbf{E}, \mathbf{B}, \rho_e, \mathbf{J}$

Maxwell's equations also split into evolution equations

[Ampere] $\partial_t \mathbf{E} = \nabla \times \mathbf{B} - 4\pi \mathbf{J} \quad \Leftrightarrow \quad \partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i$

[Faraday] $\partial_t \mathbf{B} = -\nabla \times \mathbf{E} \quad \Leftrightarrow \quad \partial_t B_i = -\epsilon_{ijk} D^j E^k$

and constraint equations [Gauss]

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e \quad \Leftrightarrow \quad \partial_i E^i = 4\pi \rho_e$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Leftrightarrow \quad \partial_i B^i = 0$$

For Maxwell's equations it is also possible to prove that if the constraint equations are satisfied at the initial time, then the evolution equations preserve that property.

To highlight even more the similarities, let us introduce the vector potential

$$A_\mu = (\Phi, A_i) \qquad \text{such that} \qquad B_i = \epsilon_{ijk} D^j A^k$$

Therefore, the evolution part of Maxwell's equations reads:

$$\begin{aligned} \partial_t A_i &= -E_i - D_i \Phi \\ \partial_t E_i &= -D^j D_j A_i + D_i D^j A_j - 4\pi J_i \end{aligned}$$

to compare with the evolution equations of the ADM system

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \\ \partial_t K_{ij} &= - \quad D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K^{kj} + K K_{ij}) \\ &\quad - \quad 8\pi\alpha (R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_\beta K_{ij}) \end{aligned}$$

Therefore, it is possible to make the following correspondence:

$$\begin{array}{ccc} \Phi & \leftrightarrow & \beta_i \\ A_i & \leftrightarrow & \gamma_{ij} \\ E_i & \leftrightarrow & K_{ij} \end{array}$$

and to note that the r.h.s. of the evolution equations of $A_i \,/\, \gamma_{ij}$ involve a field variable, $E_i \,/\, K_{ij}$,and spatial derivatives of gauge quantities $\Phi \,/\, \beta_i$

Likewise, the r.h.s. of the evolution equations of $E_i \,/\, K_{ij}$ involve matter terms and second-order spatial derivatives of the second field variable $A_i \,/\, \gamma_{ij}$

In fact, the similarities between the ADM equations and Maxwell's equations. when written in terms of a vector potential, are so close that the latter suffer from the same type of numerical problems/instabilities than the ADM equations.

To illustrate how to obtain a form of the ADM system well-suited for numerical work, let us consider again Maxwell's equations, as they are simpler and the rationale of the procedure is quite similar. Let us start with

$$\partial_t A_i = -E_i - D_i \Phi$$

We take one time derivative and use the evolution equation for the electric field E_i to obtain

$$\boxed{-\partial_t^2 A_i + D^j D_j A_i - D_i D^j A_j = D_i \partial_t \Phi - 4\pi J_i}$$

This equation would be a wave equation if the mixed derivatives term $D_i D^j A_j$ were not present.

In general relativity the situation is very similar because R_{ij} contains mixed derivatives plus a Laplacian operator acting on γ_{ij}

Without such mixed derivatives the principal part of the 3+1 ADM equations could be written as a wave equation for the 3-metric γ_{ij}

Why do we care about wave equations?

$$\Box\phi = \partial_t^2\phi - \partial^i\partial_i\phi = 0$$

Wave equations are manifestly hyperbolic and mathematical theorems guarantee existence and uniqueness of solutions.

Diverse numerical techniques have been developed to solve hyperbolic PDEs, for instance the reduction to a first-order in time, second-order in space system:

$$\Box\phi = 0 \;\Leftrightarrow\; \begin{cases} \partial_t\phi = \psi \\ \partial_t\psi = \partial^i\partial_i\phi \end{cases}$$

Stated differently: we know how to solve wave equations and what to expect.

How do we turn Maxwell's equations into a manifestly hyperbolic system?

Three ways:

1. Using a specific gauge choice, for instance the Lorentz gauge, to simplify the equations:

$$\partial_t \Phi = -D^i A_i$$

$$(\partial_t^2 - D^j D_j) A_i = \Box A_i = 4\pi J_i$$

This can also be done in general relativity by introducing harmonic coordinates and a generalized harmonic formulation of Einstein's equations.

2. Through a gauge invariant method resulting from taking a time derivative of E_i instead of A_i. This yields to

$$\partial_t^2 E_i = D_i D^j (-E_j - D_j \Phi) - D^j D_j (-E_i - D_i \Phi) - 4\pi \partial_t J_i$$

which, using the constraint equation $D^i E_i = 4\pi \rho_e$, can be written

$$\Box E_i = -4\pi (\partial_t J_i + D_i \rho_e)$$

While this is an attractive procedure it can be prone to problems since the matter source term is proportional to spatial derivatives of the charge density, $D_i \rho_e$

In general relativity this would correspond to having derivatives of the rest-mass density, and could be a divergent term in the presence of shock waves (discontinuities).

3. Introducing a new variable to remove the mixed-derivatives term, that is, by defining $\Gamma \equiv D^i A_i$ such that the evolution equation reads

$$\partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi J_i$$
$$\Box A_i = -D_i \Gamma - D_i \partial_t \Phi + 4\pi J_i$$

Clearly, it is necessary a new evolution equation for Γ which is now a new dynamical variable just like the rest (but with no physical meaning):

$$\partial_t \Gamma = \partial_t D^i A_i = D^i \partial_t A_i = -D^i E_i - D_i D^i \Phi = -4\pi \rho_e - D_i D^i \Phi$$

Despite there is one more equation, the system is now hyperbolic.

The ADM system can be modified with the introduction of new evolution variables so that the resulting system is strongly hyperbolic:

$$\phi = \frac{1}{12}\ln(\det(\gamma_{ij})) = \frac{1}{12}\ln(\gamma)$$

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$$

$$K = \gamma^{ij}K_{ij}$$

$$\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right)$$

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

ϕ conformal factor

$\tilde{\gamma}_{ij}$ conformal 3-metric

K trace of extrinsic curvature

$\tilde{A}_{ij}$ conformal, traceless extrinsic curvature

$\tilde{\Gamma}^i$ gamma's

new evolution variables

With respect to the new variables, the ADM equations read:

$$\mathcal{D}_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}$$

$$\mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t \phi = -\frac{1}{6}\alpha K$$

$$\mathcal{D}_t \tilde{A}_{ij} = e^{-4\phi}\left[-\nabla_i \nabla_j \alpha + \alpha(R_{ij} - S_{ij})\right]^{\mathrm{TF}} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j)$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i \nabla_j \alpha + \left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S)\right]$$

$$\mathcal{D}_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi\right)$$
$$-\partial_j\left(\beta^l \partial_l \tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m \beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l \beta^l\right)$$

These equations are known as BSSN equations or simply the conformal, traceless formulation of Einstein's equations.

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

Although not evident, the BSSNOK equations constitute a strongly hyperbolic system and have a structure that resembles that of a first-order in time, second-order in space system:

$$\Box\phi = 0 \;\Leftrightarrow\; \begin{cases} \partial_t \phi = \psi \\ \partial_t \psi = \partial^i \partial_i \phi \end{cases}$$ scalar wave equation

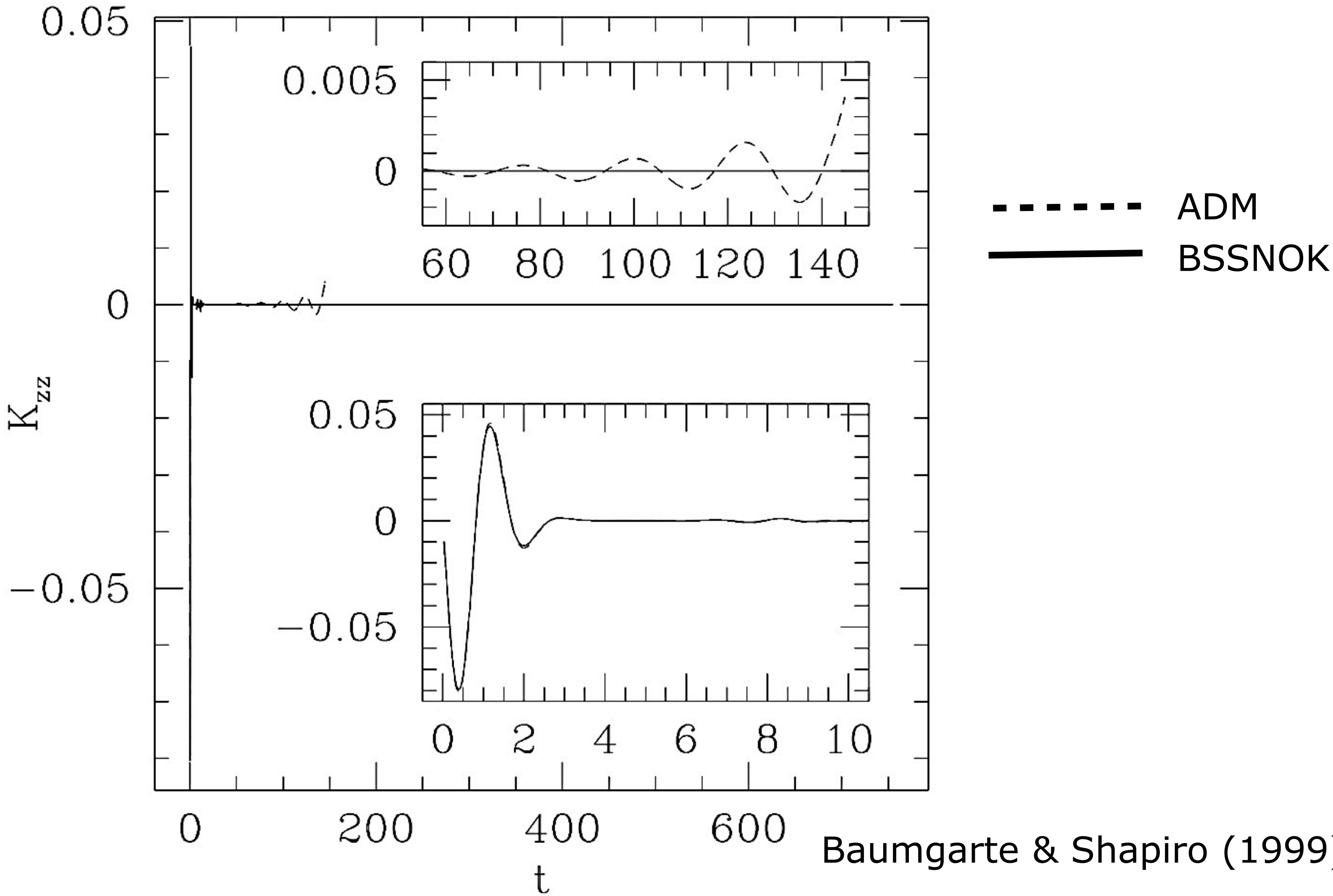
$$\begin{cases} \partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij} \\ \\ \partial_t \tilde{A}_{ij} \propto D^i D_i \tilde{\gamma}_{ij} \end{cases}$$ conformal traceless formulation

BSSNOK is nowadays the standard 3+1 formulation in NR.

Long-term stable numerical simulations have been possible for strongly gravitating systems as neutron stars (isolated and in binary systems) and black holes (isolated and in binary systems).

Example: ADM vs BSSNOK

Evolution of a gravitational wave of small amplitude

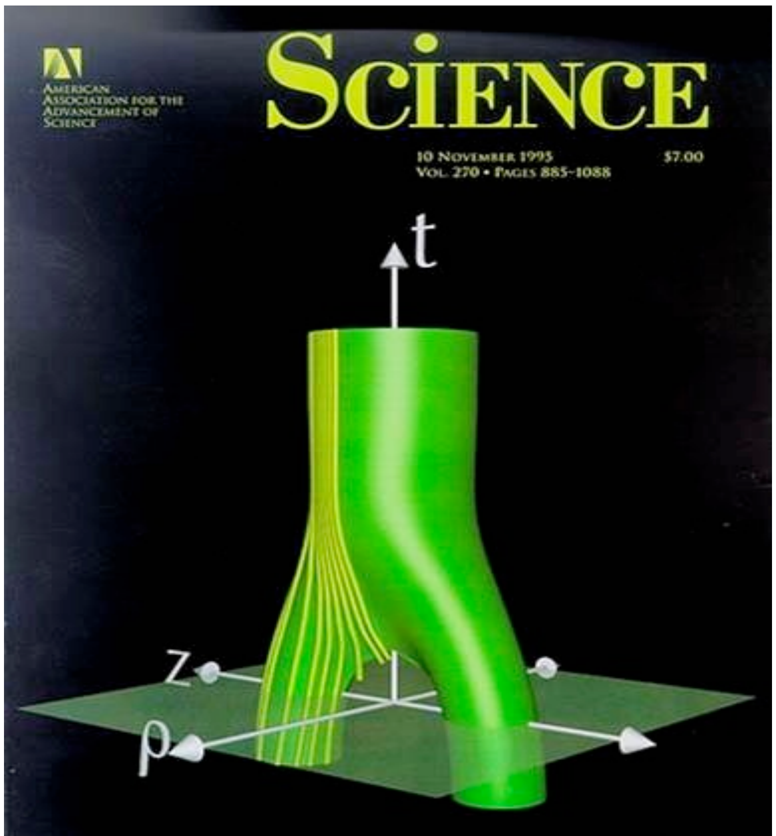


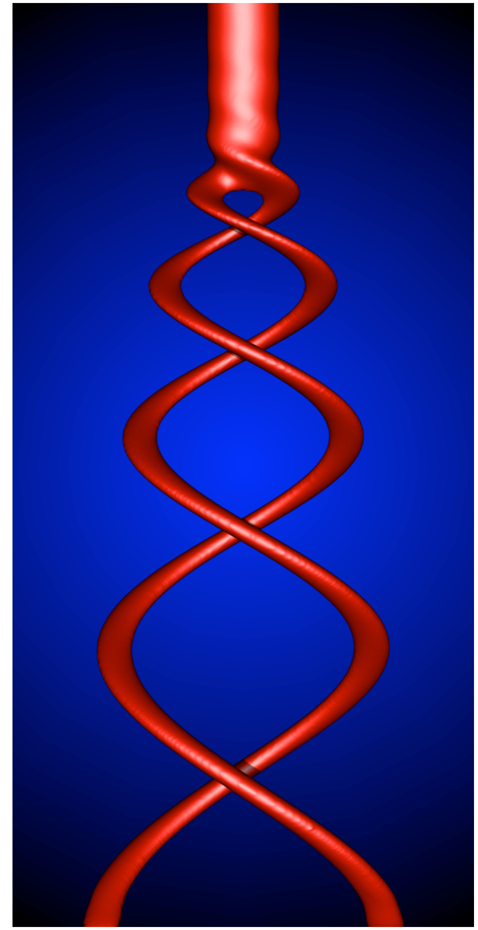
Baumgarte & Shapiro (1999)

BBH simulations: State of the art

1995: Pair of pants
(Head-on collision)

BBH Grand Challenge Alliance





FSU-Jena Numerical Relativity Group

2007: Pair of twisted pants
(spiral & merge)

Further reading

- Baumgarte & Shapiro:

  Numerical relativity: solving Einstein's equations on the computer

  (Cambridge University Press, 2010)

- Alcubierre:

  Introduction to 3+1 numerical relativity

  (Oxford University Press, 2008)

- Gourgoulhon:

  3+1 formalism and bases of numerical relativity

  (gr-qc/0703035)