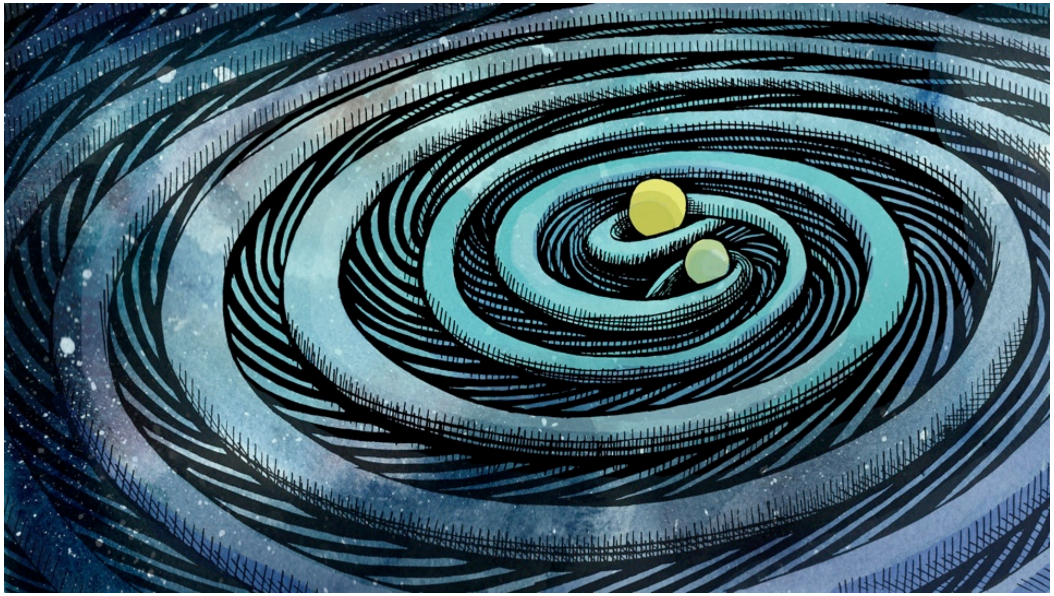
Numerical-relativity modelling of astrophysical sources of gravitational waves



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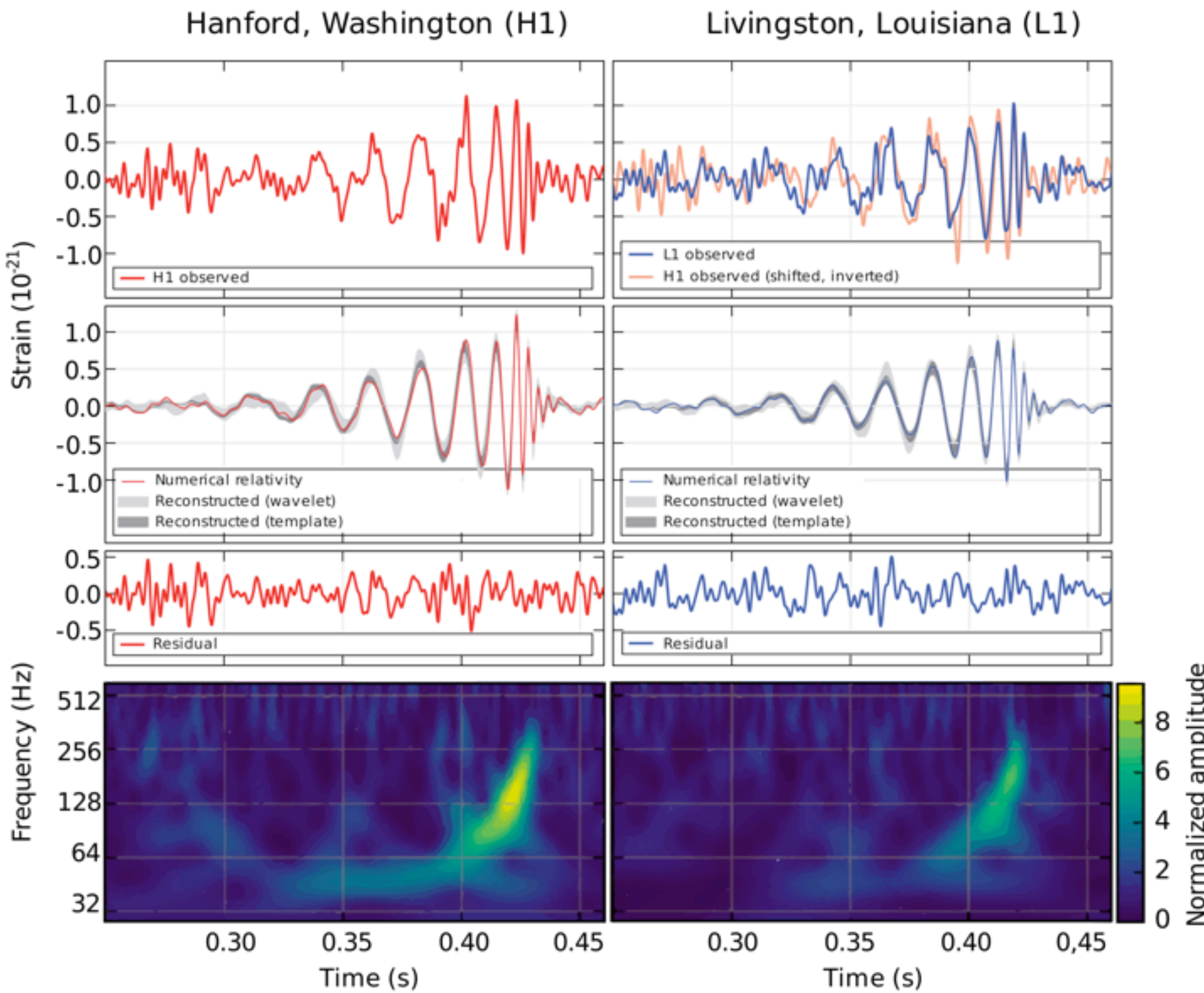
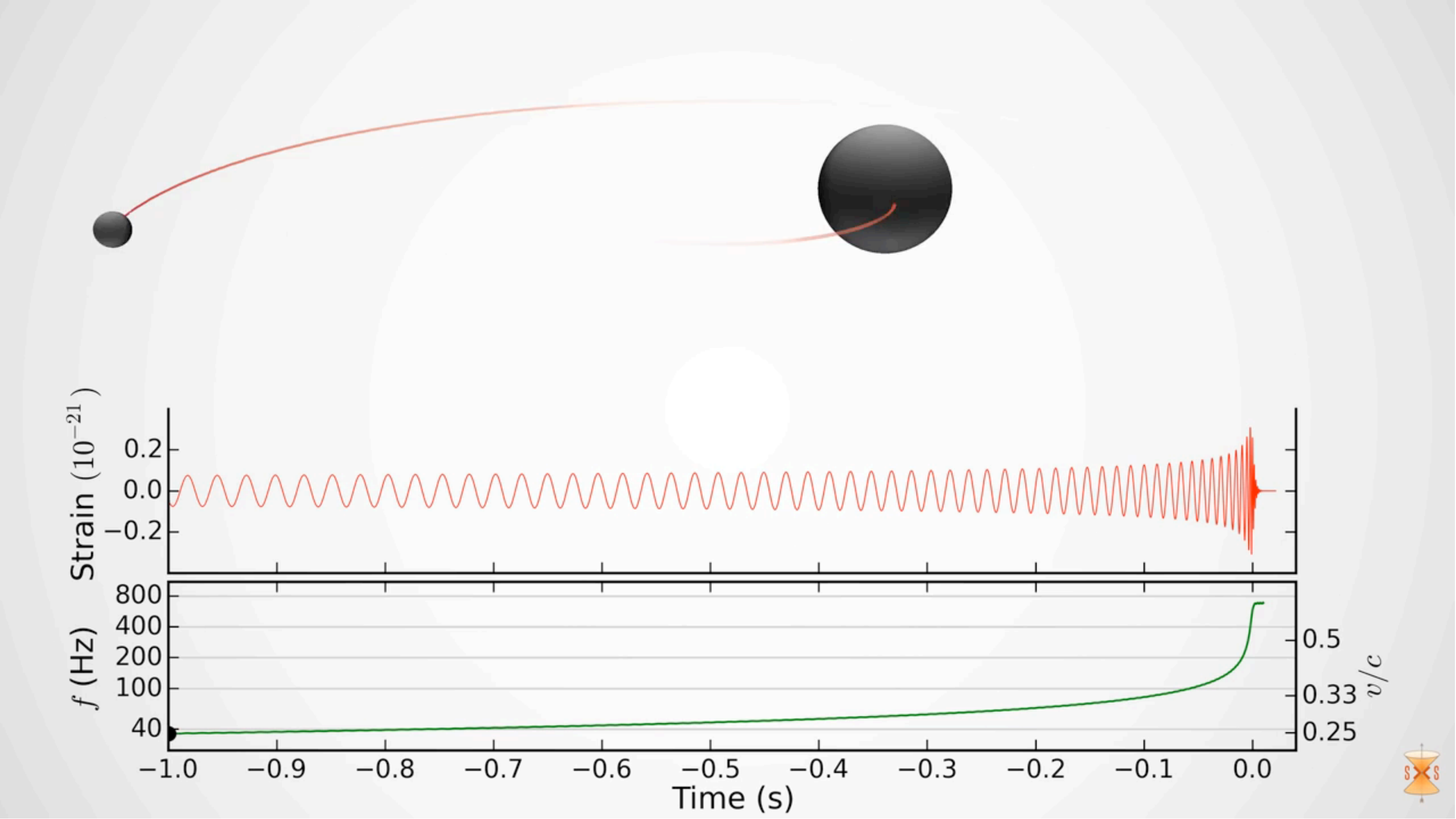
GW150914: First observation of GWs



LVC 2016

Numerical Relativity

Binary black hole merger

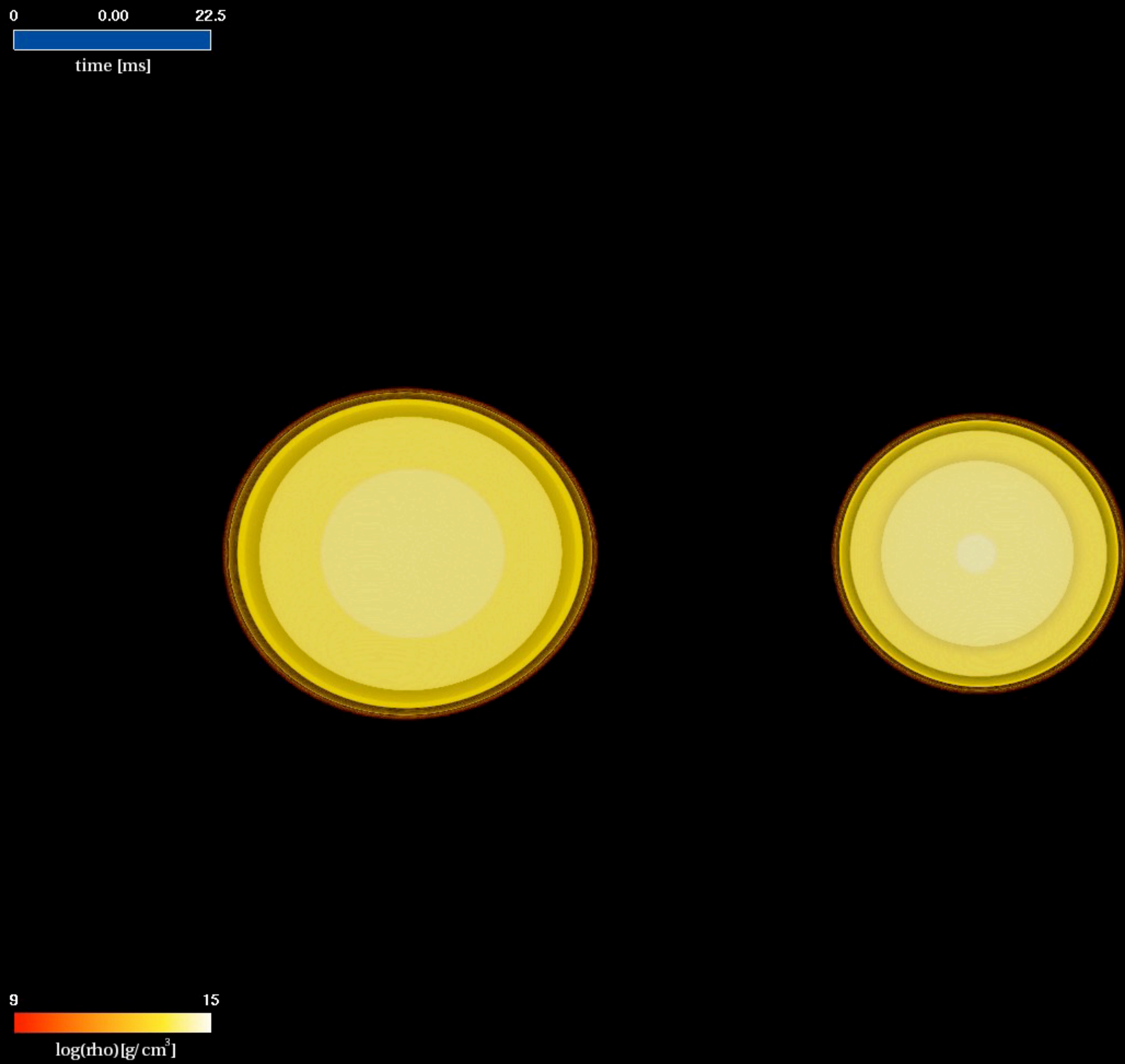


(SXS)

Neutron star binaries have been **observed** in the Galaxy.

Black hole binaries? **Hypothetical system.**

NR simulation of a BNS merger



Rezzolla+ (2011)

Our basic theoretical model

Our best approximation to the "realistic" modelling of dynamical evolutions in relativistic astrophysics (state-of-the-art?)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \qquad (\text{field eqs}: \ 6+6+3+1)$$

$$\nabla_\mu T^{\mu\nu} = 0 \ , \qquad (\text{cons. en./mom.}: \ 3+1)$$

$$\nabla_\mu (\rho u^\mu) = 0 \ , \qquad (\text{cons. of baryon no}: \ 1)$$

$$p = p(\rho, \epsilon, \ldots) \ . \qquad (\text{EoS}: \ 1 + \ldots)$$

Improvements necessary: microphysics for thermal EoS, magnetic fields, dissipative fluids, radiative transfer, ... current frontier.

$$\nabla^*_\nu F^{\mu\nu} = 0, \ (\text{Maxwell eqs.}: \ \text{induction}, \ \text{zero div.})$$

$$T_{\mu\nu} = T^{\text{fluid}}_{\mu\nu} + T^{\text{em}}_{\mu\nu} + \ldots$$

Outline

Lecture 1: Hydrodynamics and MHD

Lecture 2: Einstein's equations

Lecture 3: Numerical methods

Lecture 4: Applications in astrophysics

- Binary neutron star mergers
- Core collapse supernovae

Lecture 1
Hydrodynamics and MHD

General relativity and relativistic hydrodynamics play a major role in the description of gravitational collapse leading to the formation of compact objects (neutron stars and black holes).

Prime Sources of Gravitational Radiation.

Time-dependent evolutions of fluid flow coupled to the spacetime geometry (Einstein's equations) possible through accurate, large-scale numerical simulations.

Some scenarios can be described in the test-fluid approximation: GRHD/GRMHD computations in curved backgrounds (highly mature).

GRHD/GRMHD equations are nonlinear hyperbolic systems. Solid mathematical foundations and accurate numerical methodology imported from CFD. A "preferred" choice: high-resolution shock-capturing schemes written in conservation form. (see Lecture 3)

Fluid dynamics

The defining property of fluids (liquids and gases) lies in the ease with which they may be deformed.

A "simple fluid" may be defined as a material such that the relative positions of its constituent elements change by a large amount when suitable forces, however small in magnitude, are applied to the material.

For most simple molecules, stable equilibrium between two molecules is achieved when their separation $d_0 \sim 3\text{-}4 \times 10^{-8}$ cm.

Average spacing for gases $\sim 10\ d_0$, while in liquids and solids is $\sim d_0$.

Fluid dynamics deals with the **behaviour of matter in the large** (average quantities per unit volume), on a macroscopic scale large compared with the distance between molecules, $l \gg d_0$, not taking into account the molecular structure of fluids.

Macroscopic behaviour of fluids assumed to be continuous in structure, and physical quantities such as mass, density, or momentum contained within a given small volume are regarded as uniformly spread over that volume.

The quantities that characterize a fluid (in the continuum limit) are functions of time and position:

$$\rho \ : \ (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \rho(t,\vec{r}) \in \mathbb{R}$$ density (scalar field)

$$\vec{v} \ : \ (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \vec{v}(t,\vec{r}) \in \mathbb{R}^3$$ velocity (vector field)

$$\Pi \ : \ (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \Pi(t,\vec{r}) \in \mathbb{R}^9$$ pressure tensor (tensor field)

Eulerian description: time variation of fluid properties in a fixed position in space.

Lagrangian description: variation of properties of a “fluid particle” along its motion.

Both descriptions are equivalent: there exists a change of variables between them which is related to the Jacobian of the so-called “flux function” which describes the trajectories of fluid particles.

Transport theorems (Reynolds-Leibniz)

Generalization to 3D of the Leibniz integral rule (differentiation under the integral sign).

- Scalar field

$$\frac{d}{dt}\int_{V_t} f\,dV \;=\; \int_{V_t}\left[\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v})\right] dV, \qquad f = f(t,\vec{r})$$

- Vector field

$$\frac{d}{dt}\int_{V_t} \vec{F}\,dV \;=\; \int_{V_t}\left[\frac{\partial \vec{F}}{\partial t} + (\vec{v}\cdot\nabla)\vec{F} + \vec{F}(\nabla\cdot\vec{v})\right] dV, \quad \vec{F} = \vec{F}(t,\vec{r})$$

V_t is a volume which moves with the fluid (Lagrangian description; image of V_0 by the diffeomorphism given by the flux function).

Mass conservation (continuity equation)

Let V_t be a volume which moves with the fluid; its **mass** is given by:

$$m(V_t) \; = \; \int_{V_t} \rho(t,\vec{r})\, dV$$

Principle of conservation of mass enclosed within that volume:

$$\frac{d}{dt} m(V_t) \; = \; \frac{d}{dt} \int_{V_t} \rho(t,\vec{r})\, dV = 0$$

Applying the transport theorem for the density (scalar field):

$$0 \; = \; \frac{d}{dt} \int_{V_t} \rho(t,\vec{r})\, dV = \int_{V_t} \left[\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) \right] dV$$

where the convective derivative is defined as $\dfrac{D}{Dt} \; \equiv \; \dfrac{\partial}{\partial t} + \vec{v} \cdot \nabla$

Since the previous equation must hold for any volume V_t we obtain the **continuity equation**:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) = 0 \Rightarrow \frac{D\log\rho}{Dt} = -\Theta \Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0}$$

Corolary:

$$-\frac{\partial}{\partial t}\int_V \rho\, dV = \int_{\partial V} \rho\vec{v} \cdot d\vec{\Sigma}$$

the variation of the mass enclosed in a fixed volume V is equal to the flux of mass across the surface at the boundary of the volume.

Incompressible fluid: $\nabla \cdot \vec{v} = 0 \Leftrightarrow \frac{D\rho}{Dt} = 0$

Momentum balance (Euler's equation)

"the variation of momentum of a given portion of fluid is equal to the net force (stresses plus external forces) exerted on it" (Newton's 2nd law):

$$\frac{d}{dt}\int_{V_t} \rho\vec{v}\, dV = -\int_{\partial V_t} p\, d\vec{\Sigma} + \int_{V_t} \vec{G}\, dV = \int_{V_t} [\vec{G} - \nabla p]\, dV$$

Applying the transport theorem on the l.h.s. of the above equation:

$$\int_{V_t} \left[\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v})\right] dV = \int_{V_t} [\vec{G} - \nabla p]\, dV$$

which must be valid for any volume V_t, hence:

$$\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v}) = \vec{G} - \nabla p$$

After some algebra and using the continuity eq. we obtain **Euler's eq.**:

$$\boxed{\rho\frac{D\vec{v}}{Dt} = \vec{G} - \nabla p \;\Leftrightarrow\; \rho\vec{a} = \vec{G} - \nabla p}$$

Energy conservation

Let E be the total energy of the fluid, kinetic + internal energy:

$$E = E_{\mathrm{K}} + E_{\mathrm{int}} = \frac{1}{2}\int_{V_t} \rho \vec{v}^2\, dV + \int_{V_t} \rho\varepsilon\, dV$$

Principle of energy conservation: “the variation in time of the total energy of a portion of fluid is equal to the work done per unit time over the system by the stresses (internal forces) and the external forces”.

$$\frac{dE}{dt} = \frac{d}{dt}\int_{V_t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) dV = -\int_{\partial V_t} p\vec{v}\cdot d\vec{\Sigma} + \int_{V_t} \vec{G}\cdot\vec{v}\, dV$$

After some algebra (transport theorem, divergence theorem) we obtain:

$$\int_{V_t}\left(\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right]\right) dV = \int_{V_t} \rho\vec{g}\cdot\vec{v}\, dV \qquad \vec{g} = \frac{\vec{G}}{\rho}$$

which, as must be satisfied for any given volume, implies:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right] = \rho\vec{g}\cdot\vec{v}}$$

Hyperbolic system of conservation laws

The equations of perfect fluid dynamics are a nonlinear hyperbolic system of conservation laws:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}^i}{\partial x^i} = \vec{s}(\vec{u})}$$

state vector $\quad \vec{u} = (\rho, \rho v^j, e)$

fluxes $\quad \vec{f}^i = (\rho v^i, \rho v^i v^j + p\delta^{ij}, (e+p)v^i)$

source terms $\quad \vec{s} = \left(0, -\rho\frac{\partial \Phi}{\partial x^j}, -\rho v^i \frac{\partial \Phi}{\partial x^i}\right)$

$\vec{g}$ is a conservative external force field (e.g. gravitational field):

$$\vec{g} = -\nabla \Phi \qquad \Delta \Phi = 4\pi G \rho$$

Hyperbolic system of conservation laws

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\text{state vector} \quad \vec{u} &= (\rho, \rho v^j, e) \\
\text{fluxes} \quad \vec{f}^i &= (\rho v^i, \rho v^i v^j + p\delta^{ij}, (e+p)v^i) \\
\text{source terms} \quad \vec{s} &= \left(0, -\rho\frac{\partial \Phi}{\partial x^j} + \underline{Q_M^j}, -\rho v^i \frac{\partial \Phi}{\partial x^i} + \underline{Q_E + v^i Q_M^i}\right)
\end{aligned}$$

Q_M^i, Q_E source terms in the momentum and energy equations due to coupling between matter and radiation (when transport phenomena are taken into account).

Hyperbolic equations have finite propagation speed: information can travel with limited speed, at most that given by the largest **characteristic curves** of the system.

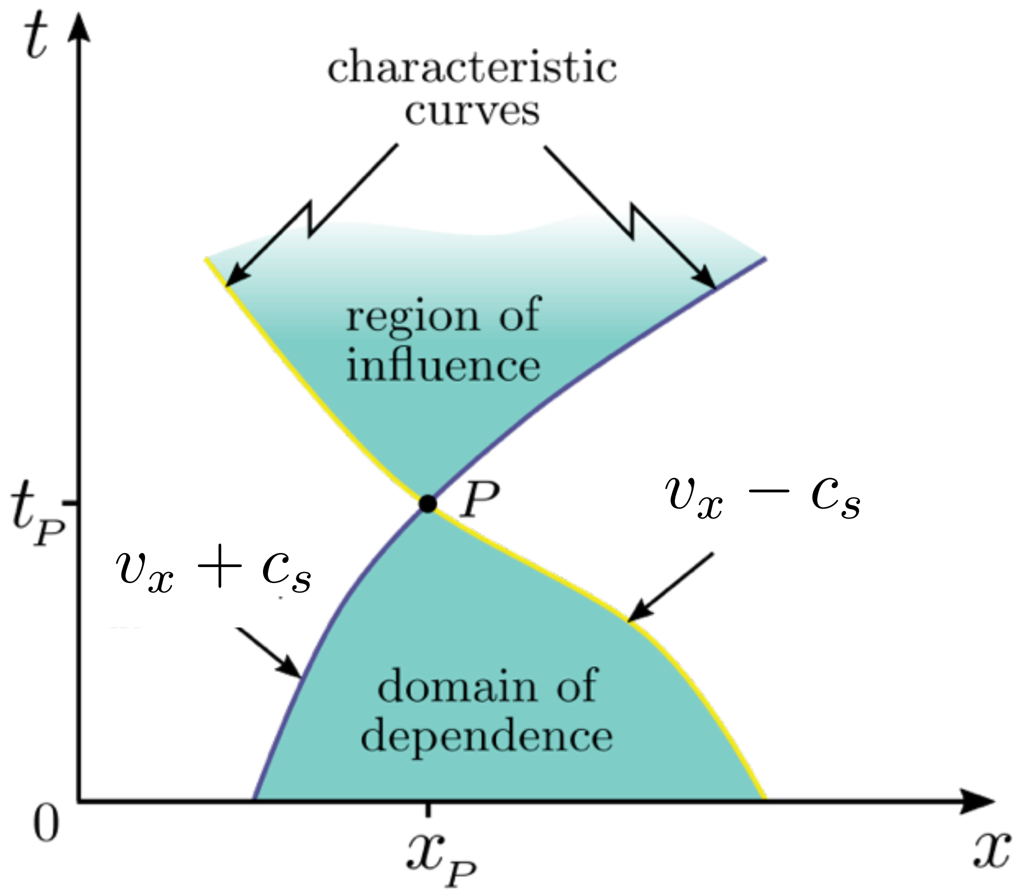
The region of influence of the solution is bounded by the eigenvalues of the Jacobian matrix of the system.

$$A = \frac{\partial \vec{f}^i}{\partial \vec{u}}$$

$$\lambda_0 = v_i$$

$$\lambda_\pm = v_i \pm c_s$$

(link with numerical schemes in Lecture 3)



A bit on viscous fluids

A perfect fluid can be defined as that for which the force across the surface separating two fluid particles is normal to that surface.

Kinetic theory tells us that the existence of velocity gradients implies the appearance of a force tangent to the surface separating two fluid layers (across which there is molecular difussion).

$$d\vec{F} = -p\,d\vec{\Sigma} \Rightarrow \boxed{d\vec{F} = -\Pi\, d\vec{\Sigma}}$$

where Π is the pressure tensor which depends on pressure and velocity gradients.

$\Pi = p\mathrm{I} - \mathrm{S}$ where S is the **stress tensor** given by:

$$\mathcal{S} = 2\mu \underbrace{\left(D - \frac{1}{3}\Theta\mathbb{I} \right)}_{\text{distortion}} + \xi \underbrace{\Theta\mathbb{I}}_{\text{expansion}}$$

shear and bulk viscosities

Using the pressure tensor in the previous derivation of the Euler eq. and of the energy eq. yields their viscous versions:

$$\rho \frac{D\vec{v}}{Dt} = \vec{G} - \nabla p + \mu \Delta \vec{v} + \left(\xi + \frac{1}{3}\mu \right) \nabla \cdot (\nabla \cdot \vec{v})$$ Navier-Stokes eq.

$$\rho \frac{D\left(\frac{1}{2}\vec{v}^2 + \varepsilon\right)}{Dt} = \rho \vec{g} \cdot \vec{v} - \nabla \cdot (p\vec{v}) + \nabla \cdot (\mathcal{S} \cdot \vec{v}) - \nabla \cdot \vec{Q}$$ Energy eq.

General relativistic hydrodynamics

The general relativistic hydrodynamics equations are obtained from the **local conservation laws of the stress-energy tensor**, $T^{\mu\nu}$ (the Bianchi identities), **and of the matter current density** J^{μ} (the continuity equation):

$$\nabla_{\mu}(\rho u^{\mu}) = 0 \qquad \nabla_{\mu} T^{\mu\nu} = 0$$

Equations of motion

$(\mu = 0, \cdots, 3)$

∇_{μ} covariant derivative associated with the four dimensional spacetime metric $g_{\mu\nu}$

The density current is given by $J^{\mu} = \rho u^{\mu}$

u^{μ} is the fluid 4-velocity and ρ is the rest-mass density in a locally inertial reference frame.

The stress-energy tensor for a **non-perfect fluid** is defined as:

$$T^{\mu\nu} = \rho(1+\varepsilon)u^{\mu}u^{\nu} + (p - \mu\Theta)h^{\mu\nu} - 2\xi\sigma^{\mu\nu} + q^{\mu}u^{\nu} + q^{\nu}u^{\mu}$$

where ε is the specific internal energy density of the fluid, p is the pressure, and $h^{\mu\nu}$ is the spatial projection tensor, $h^{\mu\nu}=u^{\mu}u^{\nu}+g^{\mu\nu}$.

In addition, μ and ξ are the shear and bulk viscosity coefficients.

The expansion, Θ, describing the divergence or convergence of the fluid world lines is defined as $\Theta=\nabla_{\mu}u^{\nu}$. The symmetric, trace-free, and spatial shear tensor $\sigma^{\mu\nu}$ is defined by:

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla_{\alpha}u^{\mu}h^{\alpha\nu} + \nabla_{\alpha}u^{\nu}h^{\alpha\mu}) - \frac{1}{3}\Theta h^{\mu\nu}$$

Finally q^{μ} is the energy flux vector.

In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, assuming the stress-energy tensor to be that of a **perfect fluid:**

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

where we have introduced the relativistic specific enthalpy, h

$$h = 1 + \varepsilon + \frac{p}{\rho}$$

Conservation laws with respect to an explicit coordinate chart

$$x^\mu = (x^0, x^i)$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\ \frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \end{aligned}}$$

Scalar x^0 represents a foliation of spacetime with hypersurfaces (with coordinates x^i), $g=\det(g_{\mu\nu})$, and $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols.

The system formed by the eqs of motion and the continuity eq must be supplemented with an **equation of state** (EOS) relating the pressure to some fundamental thermodynamical quantities, e.g.

$$p = p(\rho, \varepsilon)$$

Perfect fluid: $p = (\Gamma - 1)\rho\varepsilon$

Polytrope: $p = \kappa\rho^{\Gamma}\,, \quad \Gamma = 1 + \frac{1}{N}$

In the "test-fluid" approximation (fluid's self-gravity neglected), the dynamics of the matter fields is fully described by the previous conservation laws and the EOS. If such approximation does not hold, the previous equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime:

$$\nabla_\mu(\rho u^\mu) = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$p = p(\rho, \varepsilon)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Einstein's equations

(details on Lecture 2)

(Newtonian analogy: Euler's equation + Poisson's equation)

3+1 GR Hydro equations: formulations

$$
\begin{aligned}
\frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\
\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda}
\end{aligned}
$$

Different formulations exist depending on:

1. Choice of slicing: level surfaces of x^0 spatial (3+1) or null
2. Choice of physical (primitive) variables (ρ, ε, u^i …)

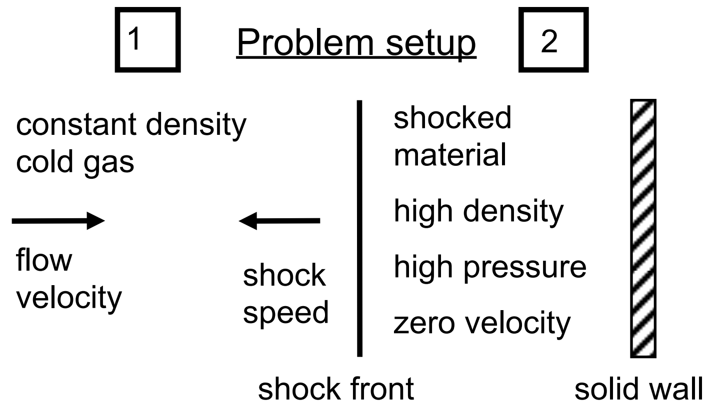
Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology are more recent:

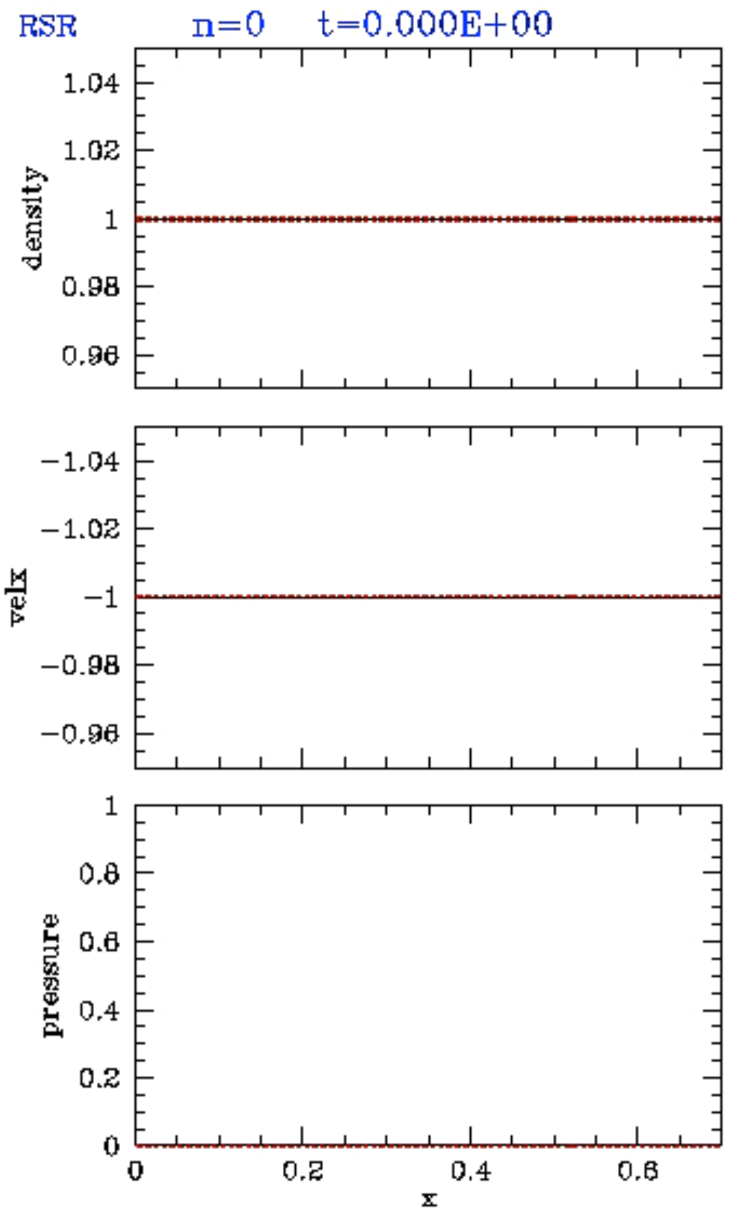
- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS ("Valencia formulation")
- Papadopoulos & Font (2000): covariant, general EOS

Relativistic shock reflection

The relativistic shock reflection problem is a demanding test involving the heating of a cold gas which impacts at relativistic speed with a solid wall creating a shock which propagates off the wall.



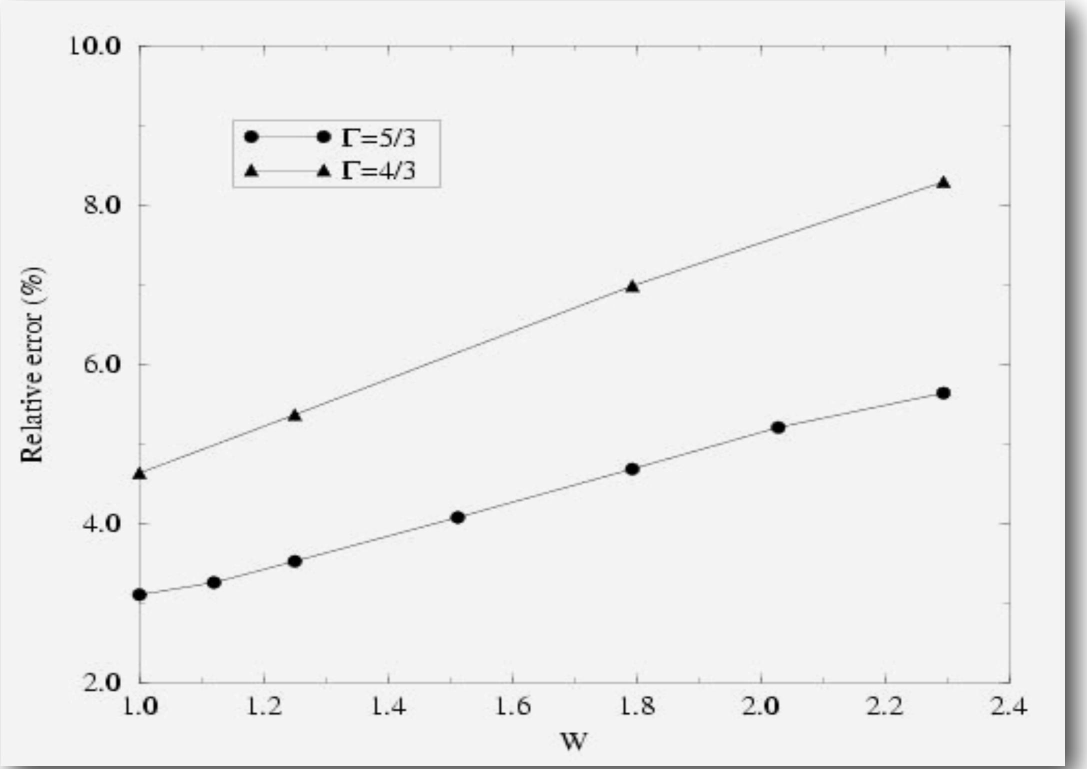
Analytic solution: $\dfrac{\rho_2}{\rho_1} = \dfrac{\gamma + 1}{\gamma - 1} + \dfrac{\gamma}{\gamma - 1}\varepsilon_2, \quad \varepsilon_2 = W_1 - 1$



(Martí & Müller, 2003)

Non-conservative formulations show limitations when simulating ultrarelativistic flows (Centrella & Wilson 1984, Norman & Winkler 1986).

Relativistic shock reflection test relative errors as a function of the fluid's Lorentz factor W. For W≈2 (v≈0.86c), error ~ 5-7% (depends on the adiabatic index of the EOS) and shows a linear increase with W.



Ultrarelativistic flows could only be handled once conservative formulations were adopted (Martí, Ibáñez & Miralles 1991; Marquina et al 1992)

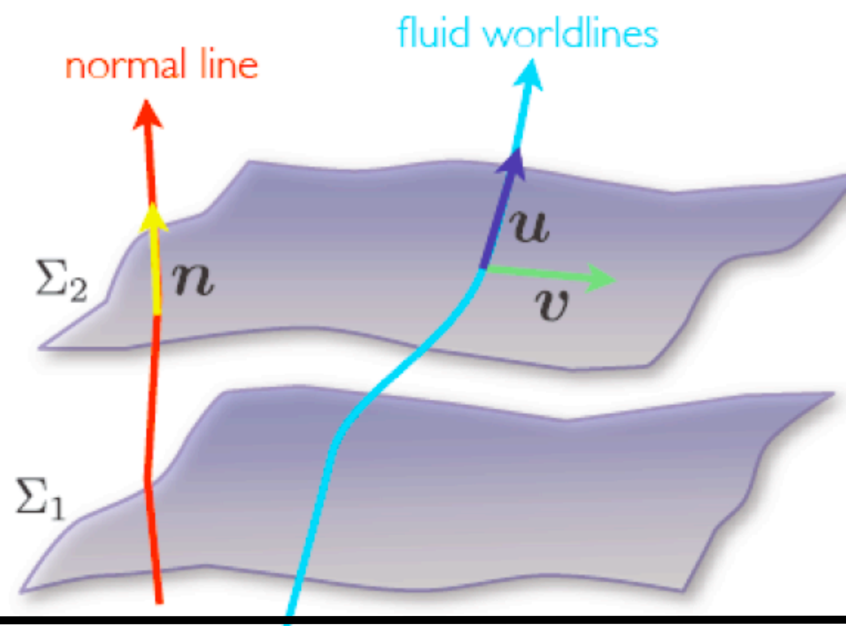
Valencia's conservative formulation (Banyuls et al 1997)

Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system, translating to relativistic hydrodynamics existing tools of CFD.

3+1: foliation of spacetime with spatial hypersurfaces Σ_t with constant t.

Line element:

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



Eulerian observer: at rest on the hypersurface; moves from Σ_t to $\Sigma_{t+\Delta t}$ along the unit normal vector. Speed given by:

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

Hyperbolic system:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial x^0}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

$$\begin{aligned}
\mathbf{U} &= (D, S_j, \tau) \\
\mathbf{F}^i &= \left(D\left(v^i-\frac{\beta^i}{\alpha}\right), S_j\left(v^i-\frac{\beta^i}{\alpha}\right)+p\delta^i_j, \tau\left(v^i-\frac{\beta^i}{\alpha}\right)+pv^i\right) \\
\mathbf{S} &= \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right)
\end{aligned}$$

First-order flux-conservative hyperbolic system

$$\begin{aligned}
D &= \rho W \\
S_j &= \rho h W^2 v_j \\
\tau &= \rho h W^2 - p - D
\end{aligned}$$

$$W^2=\frac{1}{1-v^j v_j}$$

Lorentz factor

$$h = 1+\varepsilon+\frac{p}{\rho}$$

specific enthalpy

Recovering special relativistic and Newtonian limits

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho W}{\partial t}+\frac{\partial\sqrt{-g}\rho W v^i}{\partial x^i}\right)=0$$
$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho h W^2 v^j}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}\right)=T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right)$$
$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2-\rho W)v^i}{\partial x^i}\right)=\alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)$$

Full GR

$\Downarrow$

Minkowski

$$\frac{\partial\rho W}{\partial t}+\frac{\partial\rho W v^i}{\partial x^i}=0$$
$$\frac{\partial\rho h W^2 v^j}{\partial t}+\frac{\partial(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}=0$$
$$\frac{\partial(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial(\rho h W^2-\rho W)v^i}{\partial x^i}=0$$

Newton

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v^i}{\partial x^i}=0$$
$$\frac{\partial\rho v^j}{\partial t}+\frac{\partial(\rho v^i v^j+p\delta^{ij})}{\partial x^i}=0$$
$$\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2+p\right)v^i}{\partial x^i}=0$$

(quiz: prove it!)

Eigenvalues (characteristic speeds)

Numerical schemes based on Riemann solvers use the local characteristic structure of the hyperbolic system of equations.

The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exists. The above system satisfies, hence, the definition of hyperbolicity.

Eigenvalues (along the x direction)

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)}$$

$$\lambda_\pm = \frac{\alpha}{1 - v^2 c_s^2} \left\{ v^x (1 - c_s^2) \pm c_s \sqrt{(1 - v^2)[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)]} \right\} - \beta^x$$

Special relativistic limit (along x-direction)

$$\lambda_0 = v^x \quad \text{(triple)}$$

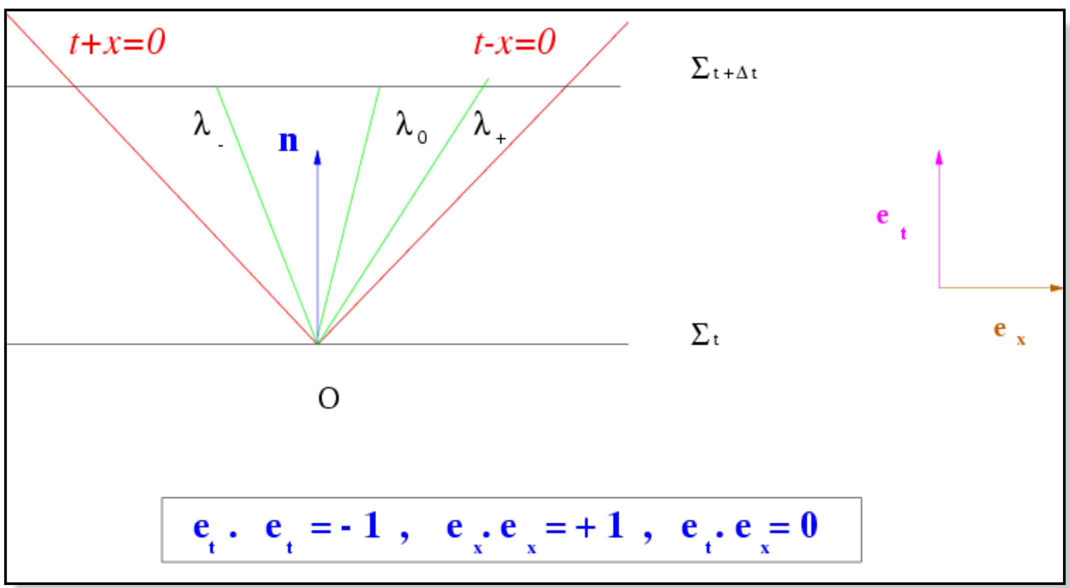
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}\left(v^x \left(1 - c_s^2\right) \pm c_s \sqrt{\left(1 - v^2\right)\left[1 - v^x v^x - \left(v^2 - v^x v^x\right) c_s^2\right]}\right)$$

coupling with transversal components of the velocity (important difference with Newtonian case)

Even in the purely 1D case:

$$\vec{v} = (v^x, 0, 0) \Rightarrow \lambda_0 = v^x, \; \lambda_\pm = \frac{v^x \pm c_s}{1 \pm v^x c_s}$$

For causal EOS sound cone lies within light cone



$\Updownarrow$

Recall Newtonian (1D) case:

$$\lambda_0 = v^x, \; \lambda_\pm = v^x \pm c_s$$

General relativistic MHD

Dynamics of relativistic, electrically conducting fluids in the presence of magnetic fields.

Ideal GRMHD: Absence of viscosity effects and heat conduction in the limit of infinite conductivity (perfect conductor fluid).

The stress-energy tensor includes contribution from the perfect fluid and from the magnetic field b^μ measured by observer comoving with the fluid.

$$T^{\mu\nu} = T^{\mu\nu}_{\rm PF} + T^{\mu\nu}_{\rm EM} \longrightarrow \boxed{T^{\mu\nu} = \rho h^* u^\mu u^\nu + p^* g^{\mu\nu} - b^\mu b^\nu}$$

$$T^{\mu\nu}_{\rm PF} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

with the definitions:

$$T^{\mu\nu}_{\rm EM} = F^{\mu\lambda}F^\nu_\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\delta}F_{\lambda\delta} = \left(u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}\right)b^2 - b^\mu b^\nu$$

$$\begin{aligned} b^2 &= b^\nu b_\nu \\ p^* &= p + \frac{b^2}{2} \\ h^* &= h + \frac{b^2}{\rho} \end{aligned}$$

$$F^{\mu\nu} = -\eta^{\mu\nu\lambda\delta}u_\lambda b_\delta$$

Ideal MHD condition: electric four-current must be finite.

$$F^{\mu\nu}u_\nu = 0$$

$$J^\mu = \rho_{\rm q} u^\mu + \sigma F^{\mu\nu}u_\nu \qquad \sigma \rightarrow \infty$$

General relativistic MHD: equations

Antón et al. (2006)

Conservation of mass: $\nabla_\mu(\rho u^\mu) = 0$

Conservation of energy and momentum: $\nabla_\mu T^{\mu\nu} = 0$

Maxwell's equations: $\nabla_\mu {}^*F^{\mu\nu} = 0$ $\quad {}^*F^{\mu\nu} = \frac{1}{W}(u^\mu B^\nu - u^\nu B^\mu)$

- Divergence-free constraint: $\vec{\nabla} \cdot \vec{B} = 0$
- Induction equation: $\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\vec{B}\right) = \vec{\nabla} \times \left[\left(\alpha\vec{v} - \vec{\beta}\right) \times \vec{B}\right]$

Adding all up: first-order, flux-conservative, hyperbolic system + constraint

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t} + \frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right) = \mathbf{S} \qquad \frac{\partial(\sqrt{\gamma}B^i)}{\partial x^i} = 0$$

$$D = \rho W \qquad S_j = \rho h^* W^2 v_j - \alpha b_j b^0 \qquad \tau = \rho h^* W^2 - p^* - \alpha^2 (b^0)^2 - D$$

$$\mathbf{U} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^k \end{bmatrix} \quad \mathbf{F}^i = \begin{bmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i + p^*\delta^i_j - b_j B^i/W \\ \tau\tilde{v}^i + p^* v^i - \alpha b^0 B^i/W \\ \tilde{v}^i B^k - \tilde{v}^k B^i \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{bmatrix}$$

MHD equations: hyperbolic structure

Wave structure classical MHD (Brio & Wu 1988): 7 physical waves

Two ALFVEN WAVES: $\lambda_{a_\pm} \Longrightarrow \lambda_a = v_x \pm v_a$

Two FAST MAGNETOSONIC WAVES: $\lambda_{f_\pm} \Longrightarrow \lambda_{f_\pm} = v_x \pm v_f$

Two SLOW MAGNETOSONIC WAVES: $\lambda_{s_\pm} \Longrightarrow \lambda_{s_\pm} = v_x \pm v_s$

One ENTROPY WAVE: $\lambda_e \Longrightarrow \lambda_e = v_x$

$$\lambda_{f_-} \leq \lambda_{a_-} \leq \lambda_{s_-} \leq \lambda_e \leq \lambda_{s_+} \leq \lambda_{a_+} \leq \lambda_{f_+}$$

$$v_{f,s}^2 = \frac{1}{2}\left\{c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho}\right)^2 - 4\left(\frac{B_x^2}{\rho}\right)c_s^2}\right\}, \quad v_a = \sqrt{\frac{B_x^2}{\rho}}$$

Anile & Pennisi (1987), Anile (1989) (see also van Putten 1991) have studied the characteristic structure of the equations (eigenvalues, right/left eigenvectors) in the space of covariant variables (u^μ,b^μ,p,s).

Wave structure for relativistic MHD (Anile 1989): roots of the characteristic equation.

Only entropic waves and Alfvén waves are explicit.

Magnetosonic waves are given by the numerical solution of a quartic equation.

Augmented system of equations: Unphysical eigenvalues/eigenvectors (entropy & Alfvén) which must be removed numerically (Anile 1989, Komissarov 1999, Balsara 2001, Koldoba et al 2002).

Further reading

A.M. Anile, "Relativistic fluids and magneto-fluids", Cambridge University Press (1989)

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F. Banyuls et al, "Numerical 3+1 general relativistic hydrodynamics: a local characteristic approach", Astrophysical Journal, 476, 221 (1997)

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L. Rezzolla & O. Zanotti, "Relativistic hydrodynamics", Oxford University Press (2013)